

On the exceptional zeros of p -non-ordinary p -adic L -functions and a conjecture of Perrin-Riou

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ABSTRACT. Our goal in this article is to prove a form of p -adic Beilinson formula for the second derivative of the p -adic L -function associated to a newform f which is non-crystalline semistable at p at its central critical point, by expressing this quantity in terms of a p -adic (cyclotomic) regulator defined on an extended trianguline Selmer group. We also prove a two-variable version of this result for height pairings we construct by considering infinitesimal deformations afforded by a Coleman family passing through f . This, among other things, leads us to a proof of an appropriate version of Perrin-Riou's conjecture in this set up.

CONTENTS

1. Introduction	2
Acknowledgements	6
2. Selmer complexes	6
2.1. (φ, Γ) -modules	6
2.2. Relation to p -adic Hodge theory	7
2.3. Cohomology of (φ, Γ) -modules	8
2.4. Iwasawa cohomology	9
2.5. The Bloch–Kato Selmer group	10
2.6. Selmer complexes	11
2.7. Duality for Selmer complexes	13
3. p -adic height pairings	14
3.1. Construction of p -adic heights	14
3.2. Cassels–Tate pairings	15
3.3. The pairing $h_{V_p, \mathbb{D}_p}^{\text{wt}}$	16
4. Selmer complexes and p -adic heights for modular forms	16
4.1. Selmer complexes for modular forms	16
4.2. p -adic families of modular forms	19
4.3. Two-variable p -adic heights	22
4.4. Rubin-style Formulae, Part II	25
5. The two-variable Perrin-Riou logarithm	32
5.1. Perrin-Riou's logarithm: General Theory	32
5.2. Perrin-Riou's logarithm for modular forms	34
6. A p -adic Beilinson formula in two-variables	35
7. Main results	40

Key words and phrases. p -adic height pairings, Selmer complexes, p -adic L -functions.

7.1. p -adic L -functions	40
7.2. Exceptional zeros of p -non-ordinary p -adic L -functions	43
7.3. On the conjecture of Perrin-Riou	44
References	45

1. INTRODUCTION

Fix forever an odd prime p . The primary goal in this article is to address a conjecture of Mazur–Tate–Teitelbaum for a p -semistable non-ordinary eigenform f . To that end, we will prove a formula relating the p -adic Abel–Jacobi image of a Heegner cycle on a suitably defined Shimura Curve to the Beilinson–Kato element. To achieve so, we will rely on a result of Seveso and the theory of p -adic heights we develop here by considering infinitesimal deformations afforded by a Coleman family \mathbf{f} passing through f . If we specialize our results to forms of weight 2, the theory we develop in this article allows us to recover Venerucci’s results in [42], relying on the finiteness of the Tate-Shafarevich groups of elliptic curves of analytic rank at most one.

Let $r_{\text{an}}(f)$ denote the order of vanishing of the Hecke L -function $L(f, s)$ at its central critical point $s = k/2$. Some of our results here simultaneously extend [4] (where similar p -adic leading term formulas has been established in the case $r_{\text{an}}(f) = 0$ but only for the cyclotomic p -adic L -function) and [12, 42] (where only the p -ordinary case (weight $k = 2$) has been treated). Although the influence of these works on our approach here is evident, it turns out to be a rather non-trivial task to build the machine suitable for the very general set up we handle here. We believe that the framework we outline here is flexible enough to treat many other interesting examples. To give an example, our Rubin-style formula (Theorem 4.13 below) is very general and it differs from what has been utilized by Venerucci in [42].

One of our main results expresses the second derivative of the Amice–Vélu, Manin, Višik p -adic L -function $L_{p,\alpha}(f, \omega^{k/2}, s)$ (where ω is the Teichmüller character) in terms of a regulator on an extended trianguline Selmer group (as studied in [5, 6]). Along the way, under the hypothesis that $r_{\text{an}}(f)$ is odd, we establish a relation between a certain Heegner cycle on an appropriately defined Shimura curve and the Beilinson–Kato class, confirming a prediction due to Perrin-Riou. This allows¹ us to compute the order of vanishing of $L_{p,\alpha}(f, \omega^{k/2}, s)$ at $s = k/2$ when $r_{\text{an}} = 1$ and therefore obtain in this situation a strong evidence towards a conjecture of Mazur–Tate–Teitelbaum.

Before we present our main results with more precision, let us introduce our set up. Fix a positive integer N coprime to p . Define S to be the set consisting of the archimedean prime and the rational primes dividing Np . Let $f = \sum_{n=1}^{\infty} a_n q^n$ be an elliptic newform of even weight k for $\Gamma_0(Np)$ such that $a_p = p^{k/2-1}$. Denote by W_f Deligne’s p -adic representation associated to f and set $V_f = W_f(k/2)$. We call V_f the central critical twist of W_f . The two-dimensional representation V_f has coefficients in a finite extension E of \mathbb{Q}_p . Furthermore, it is unramified outside S and semi-stable at p .

¹The relation we obtain between the Heegner cycle and the Beilinson–Kato element has non-trivial content granted a suitable Gross–Zagier–Zhang formula and when the order of $L(f, s)$ at $s = k/2$ is exactly 1. See Theorem 7.9, Theorem 7.8 and Remark 7.3 for details.

Let $\mathbb{D}_{\text{st}}(V_f)$ denote Fontaine's semistable Dieudonné module associated to V_f (viewed as a continuous $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -representation). Then $\mathbb{D}_{\text{st}}(V_f)$ is a two-dimensional E -vector space equipped with a Frobenius operator φ and a monodromy N given by

$$\begin{aligned} \mathbb{D}_{\text{st}}(V_f) &= Ee_\alpha + Ee_\beta, \text{ where } \varphi(e_\alpha) = \alpha e_\alpha, \varphi(e_\beta) = \beta e_\beta, \\ N(e_\beta) &= e_\alpha, \text{ and } N(e_\alpha) = 0, \\ \beta &= 1, \text{ and } \alpha = p^{-1}. \end{aligned}$$

Let $\mathbb{D}_{\text{rig}}^\dagger(V_f)$ denote Fontaine's (étale) (φ, Γ) -module associated to V_f . We set $D := Ee_\alpha$ which is the unique non-trivial (φ, N) -submodule of $\mathbb{D}_{\text{st}}(V_f)$. Let \mathbb{D}_f denote the (φ, Γ) -submodule of $\mathbb{D}_{\text{rig}}^\dagger(V_f)$ associated to D by Berger, and $\widetilde{\mathbb{D}}_f = \mathbb{D}_{\text{rig}}^\dagger(V_f)/\mathbb{D}_f$. Both \mathbb{D}_f and $\widetilde{\mathbb{D}}_f$ are (φ, Γ) -modules of rank 1, and as such may be described rather explicitly (see Proposition 4.1). This is crucial for our calculations.

Attached to the pair (V_f, D) , one associates a *Selmer complex* $\widetilde{C}_f^\bullet(V_f) = S^\bullet(V_f, \mathbb{D}_f)$ (see Section 4.3). Let $\mathbf{R}\Gamma(V_f, \mathbb{D}_f)$ denote the associated object in the derived category and define the *extended trianguline Selmer group* $\widetilde{H}_f^1(V_f) = \mathbf{R}^1\Gamma(V_f, \mathbb{D}_f)$ as the cohomology of the Selmer complex in degree 1. We define a variety of height pairings on these extended Selmer groups by considering various infinitesimal deformations of V_f , generalizing previous constructions in this context. Among others, we have a *symmetric* (cyclotomic) p -adic height pairing

$$\mathfrak{h}_p : \widetilde{H}_f^1(V_f) \times \widetilde{H}_f^1(V_f) \longrightarrow E$$

and a *skew-symmetric* p -adic height pairing

$$\mathfrak{h}_f^{\text{c-wt}} : \widetilde{H}_f^1(V_f) \times \widetilde{H}_f^1(V_f) \longrightarrow E$$

which we call the *central critical height pairing*. See Section 4.1 for the definition of the former and Definition 4.8 for the latter. These two height pairings appear naturally as the specializations of a *two-variable* height pairing

$$\mathfrak{H}_f : \widetilde{H}_f^1(V_f) \times \widetilde{H}_f^1(V_f) \longrightarrow \mathfrak{I}/\mathfrak{I}^2$$

where $\mathfrak{I} \subset E[[\kappa - k, s]]$ is the ideal of functions those vanish at $(k, 0)$. The pairing \mathfrak{H}_f is denoted by $\mathcal{A}(\mathbb{H}_f)$ in the main text and it is introduced in Section 4.3. The two-variable height pairing \mathfrak{H}_f also satisfies an appropriate functional equation.

The extended trianguline Selmer group compares with the Bloch–Kato Selmer group $H_f^1(\mathbb{Q}, V_f)$ via the exact sequence

$$0 \longrightarrow H^0(\widetilde{\mathbb{D}}_f) \xrightarrow{\partial_0} \widetilde{H}_f^1(V_f) \longrightarrow H_f^1(\mathbb{Q}, V_f) \longrightarrow 0.$$

This exact sequence admits a natural splitting $\text{spl} : \widetilde{H}_f^1(V_f) \rightarrow H^0(\widetilde{\mathbb{D}}_f)$ (Proposition 4.2), and in our setting it turns out that the E -vector space $H^0(\widetilde{\mathbb{D}}_f)$ is one-dimensional. This is a reflection of the (simple) *exceptional zero* that the p -adic L -function $L_{p,\alpha}(f, \omega^{k/2}, s)$ possesses at the central critical value $s = k/2$.

Until the end of this introduction, suppose $L(f, k/2) = 0$. It follows from [22] (see also [14], [4]) that $\text{ord}_{s=k/2} L_{p,\alpha}(f, \omega^{k/2}, s) \geq 2$. Furthermore, Kato's work [22] equips us with an element $[z_f^{\text{BK}}] \in H_f^1(\mathbb{Q}, V_f)$, which is the first layer of the Beilinson–Kato element Euler system. We note that a natural extension of a conjecture of Perrin-Riou [33, §3.3.2] would assert that the class $[z_f^{\text{BK}}]$ is non-zero if and only if $\text{ord}_{s=k/2} L(f, s) = 1$; see Theorem D below for our result towards this prediction. We let $[\mathfrak{z}_f^{\text{BK}}] \in \widetilde{H}_f^1(V_f)$ denote

the canonical lift of the class $[z_f^{\text{BK}}]$ in $\tilde{H}_f^1(V_f)$. Let $d_{\tilde{f}}$ denote a distinguished generator of $H^0(\tilde{\mathbb{D}}_f)$ that we introduce at the very end of Section 4.1 and which depends on the choice of e_α . Finally, for a class $[x] \in H_{\mathbf{f}}^1(V_f)$ we write

$$\Omega([x]) := \left(1 - \frac{1}{p}\right) \Gamma(k/2 - 1)^{-1} \langle \Psi_2, \text{res}_p([x]) \rangle \cdot \langle \Psi_1, \partial_0(d_{\tilde{f}}) \rangle \in E.$$

Here $\{\Psi_1, \Psi_2\}$ is a distinguished basis of $H^1(\tilde{\mathbb{D}}_f)$ which is given by Definition 4.3 and the pairing $\langle \cdot, \cdot \rangle : H^1(\tilde{\mathbb{D}}_f) \times H^1(\tilde{\mathbb{D}}_f) \rightarrow E$ is the natural pairing.

We are now ready to state a sample of our main results. The first is a p -adic Beilinson formula for the second derivative of the p -adic L -function.

Theorem A. *We have*

$$\frac{\Omega \cdot [e_\alpha, b_f^*]_{V_f}}{2} \cdot \frac{d^2}{ds^2} L_{p,\alpha}(f, \omega^{k/2}, s) \Big|_{s=k/2} = \det \begin{pmatrix} \mathfrak{h}_p(\partial_0(d_{\tilde{f}}), \partial_0(d_{\tilde{f}})) & \mathfrak{h}_p(\partial_0(d_{\tilde{f}}), [z_f^{\text{BK}}]) \\ \mathfrak{h}_p([z_f^{\text{BK}}], \partial_0(d_{\tilde{f}})) & \mathfrak{h}_p([z_f^{\text{BK}}], [z_f^{\text{BK}}]) \end{pmatrix},$$

where $\Omega = \Omega([z_f^{\text{BK}}])$, b_f^* denote the basis of $\text{Fil}^0 \mathbb{D}_{\text{st}}(V_f)$ defined by Kato in [22, Theorem 12.5], and

$$[\cdot, \cdot]_{V_f} : \mathbb{D}_{\text{st}}(V_f) \times \mathbb{D}_{\text{st}}(V_f) \rightarrow E$$

the canonical duality.

Note that the submodule $D \subset \mathbb{D}_{\text{st}}(V_f)$ defines a canonical splitting of the Hodge filtration on $\mathbb{D}_{\text{st}}(V)$:

$$\mathbb{D}_{\text{st}}(V_f) = D \oplus \text{Fil}^0 \mathbb{D}_{\text{st}}(V_f).$$

We let

$$h^{\text{Nek}} : H_{\mathbf{f}}^1(\mathbb{Q}, V_f) \times H_{\mathbf{f}}^1(\mathbb{Q}, V_f) \rightarrow E$$

denote Nekovář's p -adic height pairing [30] associated to this splitting. Theorem 11 of [6] gives a precise relationship between h^{Nek} and \mathfrak{h}_p which generalizes [31, Theorem 11.4.6] to the non-ordinary case. Applying a standard argument (see, for example, the proof of [30, Theorem 7.13]) we obtain the following reformulation of Theorem A.

Corollary B. *Assume that the Fontaine–Mazur $\mathcal{L}_{\text{FM}}(f)$ does not vanish. Then*

$$\left(1 - \frac{1}{p}\right) \frac{[e_\alpha, b_f^*]_{V_f}}{2\Gamma(k/2 - 1)} \langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle \cdot \frac{d^2}{ds^2} L_{p,\alpha}(f, \omega^{k/2}, s) \Big|_{s=k/2} = \mathcal{L}_{\text{FM}}(f) \cdot h^{\text{Nek}}([z_f^{\text{BK}}], [z_f^{\text{BK}}]).$$

Theorem A is Theorem 7.5 in the main text and it in fact follows from the following leading term formula for the two-variable p -adic L -function $L_p(\mathfrak{X}, \kappa, s)$ defined in Section 5.2. Here $\mathfrak{X} \in H^1(G_{\mathbb{Q},S}, \overline{V}_{\mathbf{f}})$ and $\overline{V}_{\mathbf{f}} := V_{\mathbf{f}} \otimes_E \mathcal{H}^i$ is the cyclotomic deformation of the big Galois representation attached to a Coleman family \mathbf{f} that specializes in weight k to our eigenform f . Let $\text{pr} : H^1(G_{\mathbb{Q},S}, \overline{V}_{\mathbf{f}}) \rightarrow H^1(G_{\mathbb{Q},S}, V_f)$ denote the obvious projection.

Theorem C. *Let $[x_{\mathbf{f}}] = [(x, x_\ell, \lambda_\ell)]$ be an element such that $[x] \in H_{\mathbf{f}}^1(\mathbb{Q}, V_f)$. Suppose further that there is an element $\mathfrak{X} \in H^1(G_{\mathbb{Q},S}, \overline{V}_{\mathbf{f}})$ with the property that $\text{pr}(\mathfrak{X}) = [x]$.*

Then,

$$\Omega([x]) \cdot L_p(\mathfrak{X}, \kappa, s) \equiv \det \begin{pmatrix} \mathfrak{H}_{\mathbf{f}}(\partial_0(d_{\tilde{s}}), \partial_0(d_{\tilde{s}})) & \mathfrak{H}_{\mathbf{f}}(\partial_0(d_{\tilde{s}}), [x_f]) \\ \mathfrak{H}_{\mathbf{f}}([x_f], \partial_0(d_{\tilde{s}})) & \mathfrak{H}_{\mathbf{f}}([x_f], [x_f]) \end{pmatrix} \pmod{\mathfrak{I}^3}.$$

This is Theorem 6.2(ii) below and we may deduce Theorem A from its statement plugging in $\kappa = k$. It may also be used together with the main results of [37] to deduce the following version of Perrin-Riou's prediction in [33, §3.3.2]. Before we state our result, let us introduce the necessary notation. Let L be an algebraic number field containing all coefficients a_n of f . Let $E \supset \mathbb{Q}_{p^2}$ denote its completion at a fixed (arbitrary) prime above p . Let the $\mathcal{M}_{k-2}/\mathbb{Q}$ denote the Iovita–Spiess Chow motive of weight k modular forms (whose p -adic realisation affords representations associated to cusp forms which are new at pN^-). There exists a map

$$\log \Phi^{\text{AJ}} : \text{CH}^{k/2}(\mathcal{M}_{k-2} \otimes H) \longrightarrow M_k(\Gamma, E)^*,$$

(which is essentially identical to the p -adic Abel–Jacobi map) where

- Γ is a certain ‘congruence subgroup’ of a suitably chosen quaternion algebra,
- H is a certain extension of K ,
- $M_k(\Gamma, E)^*$ is the dual of the space of rigid analytic modular forms for Γ .
- $\text{CH}^{k/2}(\mathcal{M}_{k-2} \otimes H)$ is the Chow group of codimension $k/2$ cycles on $\mathcal{M}_{k-2} \otimes H$.

Let $y^\epsilon \in \text{CH}^{k/2}(\mathcal{M}_{k-2} \otimes H)$ denote a Heegner cycle which is given as in Theorem 7.4.

For Theorem D, we will assume that $L(f, s)$ vanishes at $s = k/2$ to odd order. This allows us to choose the character ϵ (that plays a role in the definition of the Heegner cycle, for which we refer the reader to [37]) to be the trivial character. This is our convention for the rest of this introduction.

Theorem D. *Suppose that $\log \Phi^{\text{AJ}}(y^\epsilon)|_{f_{\text{rig}}}$ is non-trivial. Then the restriction of the Beilinson–Kato class $\text{res}_p([z_f^{\text{BK}}])$ at the prime p does not vanish.*

This is Theorem 7.9 in the main body of this article. Theorem D may be used together with Theorem A in order to deduce the following evidence towards the Mazur–Tate–Teitelbaum conjecture.

Corollary E. *Assume that the (cyclotomic) p -adic height pairing \mathfrak{h}_p is non-degenerate. Under the hypothesis of Theorem D we have*

$$\text{ord}_{s=k/2} L_p(f, s) = 2.$$

We expect that the hypothesis (both on Theorem D and Corollary E) requiring the non-vanishing of $\log \Phi^{\text{AJ}}(y^\epsilon)|_{f_{\text{rig}}}$ may soon be replaced by the hypothesis that $r_{\text{an}}(f) = 1$. Indeed, a suitable Gross–Zagier–Zhang formula (which is currently unavailable in the level generality we require) would show that the non-vanishing of the Abel–Jacobi image of the Heegner cycle y^ϵ is equivalent to asking that $r_{\text{an}}(f) = 1$. This in particular would show in this situation that

$$\text{ord}_{s=k/2} L_p(f, s) = 1 + r_{\text{an}}(f),$$

as predicted by Mazur–Tate–Teitelbaum. We further remark that the p -adic Abel–Jacobi map is always expected to be injective in this set up.

Remark 1.1. Although we have not checked the details, it seems that our theory here applies equally well for p -semistable modular forms whose p -adic L -functions do *not* necessarily possess an exceptional zero and whose Hecke L -function vanish at their central critical point to order 1. This leads us to a p -adic Gross-Zagier formula which expresses the leading term of its p -adic L -function in terms of the p -adic height of the Beilinson-Kato element, allowing us to extend and generalize the results of [33] to higher weight forms.

Acknowledgements. The both authors wish to thank M. Seveso who kindly pointed to us that his work should be sufficient to complement our initial results to yield a proof of Perrin-Riou's conjecture in this context. The second named author (K.B.) wishes to thank first author for his interest in his prior work [12] and his repeated invitations to University of Bordeaux. It is during these visits that much of this research was carried out. K.B. also thanks Daniel Disegni, Olivier Fouquet, Benjamin Howard, Rodolfo Venerucci, Xinyi Yuan and Wei Zhang for helpful conversations, and Henri Darmon for inviting K.B. to a CRM workshop during which he had the chance to interact with M. Seveso.

2. SELMER COMPLEXES

2.1. (φ, Γ) -modules. In this Section we review the theory of Selmer complexes and p -adic heights following [6] and apply the general machinery to the case of p -adic representations arising from modular forms. In particular, we construct an infinitesimal deformation of the p -adic height pairing in the weight direction generalizing the work of Venerucci [42].

We start by introducing some general notation. Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of \mathbb{Q}_p , $G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and \mathbb{C}_p the p -adic completion of $\overline{\mathbb{Q}_p}$. We denote by $v_p : \mathbb{C}_p \rightarrow \mathbb{R} \cup \{+\infty\}$ the p -adic valuation on \mathbb{C}_p normalized by $v_p(p) = 1$ and set $|x|_p = \left(\frac{1}{p}\right)^{v_p(x)}$. Fix a system $\varepsilon = (\zeta_{p^n})_{n \geq 1}$ of p^n th roots of the unity such that $\zeta_p \neq 1$ and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all n . We set $\Gamma = \text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ and denote by $\chi : \Gamma \rightarrow \mathbb{Z}_p^*$ the cyclotomic character. The group Γ decomposes canonically into the direct sum $\Gamma = \Delta \times \Gamma_0$, where $\Delta = \text{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ and $\Gamma_0 = \text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p(\zeta_p))$. We denote by $\langle \chi \rangle : \Gamma \rightarrow \mathbb{Z}_p^*$ the composition of $\chi|_{\Gamma_0}$ with the canonical projection $\Gamma \rightarrow \Gamma_0$.

For each $0 \leq r < 1$ we denote by $\text{ann}(r, 1)$ the p -adic annulus

$$\text{ann}(r, 1) = \{x \in \mathbb{C}_p \mid p^{-1/r} \leq |x|_p < 1\}.$$

If E is a finite extension of \mathbb{Q}_p we denote by $\mathcal{R}_E^{(r)}$ the ring of power series $f(\pi) = \sum_{n=-\infty}^{\infty} a_n \pi^n$ with coefficients in E converging on $\text{ann}(r, 1)$. The Robba ring \mathcal{R}_E is defined to be the union $\mathcal{R}_E = \bigcup_{0 \leq r < 1} \mathcal{R}_E^{(r)}$. We equip \mathcal{R}_E with a continuous action of the group $\Gamma = \text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ and a Frobenius φ given by

$$(2.1) \quad \tau(f(\pi)) = f((1 + \pi)^{\chi(\tau)} - 1), \quad \tau \in \Gamma,$$

$$(2.2) \quad \varphi(f(\pi)) = f((1 + \pi)^p - 1).$$

More precisely, we have

$$\begin{aligned}\tau(\mathcal{R}_E^{(r)}) &= \mathcal{R}_E^{(r)}, & \tau \in \Gamma, \\ \varphi(\mathcal{R}_E^{(r)}) &= \mathcal{R}_E^{(pr)}.\end{aligned}$$

The rings $\mathcal{R}_E^{(r)}$ are equipped with a canonical Frechet topology [8]. For each affinoid E -algebra A define

$$\mathcal{R}_A^{(r)} = \mathcal{R}_E^{(r)} \widehat{\otimes}_E A, \quad \mathcal{R}_A = \bigcup_{0 \leq r < 1} \mathcal{R}_A^{(r)}.$$

The actions of φ and Γ on \mathcal{R}_E extend by linearity to \mathcal{R}_A .

Definition 2.1. i) A (φ, Γ) -module over $\mathcal{R}_A^{(r)}$ is a finitely generated projective $\mathcal{R}_A^{(r)}$ -module $\mathbb{D}^{(r)}$ equipped with the following structures:

a) A φ -semilinear map

$$\varphi : \mathbb{D}^{(r)} \rightarrow \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A^{(pr)}$$

such that the induced linear map

$$\varphi^* : \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}, \varphi} \mathcal{R}_A^{(pr)} \rightarrow \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A^{(pr)}$$

is an isomorphism of $\mathcal{R}_A^{(pr)}$ -modules.

b) A semilinear continuous action of Γ on $\mathbb{D}^{(r)}$ commuting with φ .

ii) \mathbb{D} is a (φ, Γ) -module over \mathcal{R}_A if $\mathbb{D} = \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A$ for some (φ, Γ) -module $\mathbb{D}^{(r)}$ over $\mathcal{R}_A^{(r)}$.

For each continuous character $\delta : \mathbb{Q}_p^* \rightarrow A^*$, we denote by $\mathcal{R}_A(\delta)$ the (φ, Γ) -module $\mathcal{R}_A \cdot e_\delta$ of rank 1 generated by e_δ and such that

$$\begin{aligned}\tau(e_\delta) &= \delta(\chi(\tau)) \cdot e_\delta, & \tau \in \Gamma, \\ \varphi(e_\delta) &= \delta(p) \cdot e_\delta.\end{aligned}$$

If $A = E$ is a finite extension of \mathbb{Q}_p , each (φ, Γ) -module of rank 1 over \mathcal{R}_E is isomorphic to $\mathcal{R}_E(\delta)$ for some character δ , see [15].

A p -adic representation of $G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ with coefficients in an affinoid algebra A is a finitely generated projective A -module equipped with a continuous linear action of G_p . The theory of (φ, Γ) -modules associates to each p -adic representation V of G_p a (φ, Γ) -module $\mathbb{D}_{\text{rig}, A}^\dagger(V)$ over \mathcal{R}_A . The functor $V \mapsto \mathbb{D}_{\text{rig}, A}^\dagger(V)$ is fully faithful and commutes with base change. If $A = E$, we write $\mathbb{D}_{\text{rig}}^\dagger(V)$ instead of $\mathbb{D}_{\text{rig}, E}^\dagger(V)$ to simplify notation. A theorem of Kedlaya proved in [23] implies that the functor $\mathbb{D}_{\text{rig}}^\dagger$ establishes an equivalence between the category of p -adic representations with coefficients in E and the category of (φ, Γ) -modules over \mathcal{R}_E of slope 0.

For each (φ, Γ) -module \mathbb{D} we denote by \mathbb{D}^* its dual module $\mathbb{D}^* = \text{Hom}_{\mathcal{R}_A}(\mathbb{D}, \mathcal{R}_A)$. The twist $\mathbb{D}^*(\chi)$ of \mathbb{D}^* by the cyclotomic character is the Tate dual of \mathbb{D} .

2.2. Relation to p -adic Hodge theory. A filtered φ -module (resp. a filtered (φ, N) -module) is a finite-dimensional E -vector space M equipped with an exhaustive decreasing filtration $(\text{Fil}^i M)_{i \in \mathbb{Z}}$ and a bijective Frobenius $\varphi : M \rightarrow M$ (resp. a bijective Frobenius φ and a nilpotent monodromy $N : M \rightarrow M$ such that $N\varphi = p\varphi N$). In [10], Berger

associated to each (φ, Γ) -module \mathbb{D} a filtered (φ, N) -module $\mathcal{D}_{\text{st}}(\mathbb{D})$ and a filtered φ -module $\mathcal{D}_{\text{cris}}(\mathbb{D})$ such that $\mathcal{D}_{\text{cris}}(\mathbb{D}) = \mathcal{D}_{\text{st}}(\mathbb{D})^{N=0}$. Moreover

$$\dim_E \mathcal{D}_{\text{cris}}(\mathbb{D}) \leq \dim_E \mathcal{D}_{\text{st}}(\mathbb{D}) \leq \text{rank}_{\mathcal{R}_E}(\mathbb{D}).$$

We say that a (φ, Γ) -module \mathbb{D} is semistable (resp. crystalline) if we have $\dim_E \mathcal{D}_{\text{st}}(\mathbb{D}) = \text{rank}_{\mathcal{R}_E}(\mathbb{D})$ (resp. if $\dim_E \mathcal{D}_{\text{cris}}(\mathbb{D}) = \text{rank}_{\mathcal{R}_E}(\mathbb{D})$). The functor $\mathbb{D} \mapsto \mathcal{D}_{\text{cris}}(\mathbb{D})$ (resp. $\mathbb{D} \mapsto \mathcal{D}_{\text{st}}(\mathbb{D})$) establishes an equivalence between the category of crystalline (resp. semistable) (φ, Γ) -modules and the category of filtered φ -modules (resp. (φ, N) -modules) (see [10]). If V is a p -adic representation with coefficients in E , then we have canonical and functorial isomorphisms

$$\mathbb{D}_{\text{st}}(V) = \mathcal{D}_{\text{st}}(\mathbb{D}_{\text{rig}}^\dagger(V)), \quad \mathbb{D}_{\text{cris}}(V) = \mathcal{D}_{\text{cris}}(\mathbb{D}_{\text{rig}}^\dagger(V)),$$

where $\mathbb{D}_{\text{st}}(V)$ and \mathbb{D}_{cris} are the classical Fontaine functors.

2.3. Cohomology of (φ, Γ) -modules. We review the cohomology of (φ, Γ) -modules. Let $\Gamma = \Delta \times \Gamma_0$ be the canonical decomposition of Γ into the direct product of $\Delta = \text{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ and the pro- p -cyclic group $\Gamma_0 = \text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p(\zeta_p))$. Fix a generator $\gamma \in \Gamma_0$. Let A be an affinoid algebra over \mathcal{R}_A . For any (φ, Γ) -module \mathbb{D} over \mathcal{R}_A we can consider the Fontaine–Herr complex

$$C_{\varphi, \gamma}^\bullet(\mathbb{D}) : \mathbb{D}^\Delta \xrightarrow{d_0} \mathbb{D}^\Delta \oplus \mathbb{D}^\Delta \xrightarrow{d_1} \mathbb{D}^\Delta,$$

where $d_0(x) = ((\varphi - 1)(x), (\gamma - 1)x)$ and $d_1(y, z) = (\gamma - 1)(y) - (\varphi - 1)(z)$ (see [20], [25], [24]). Define

$$H^i(\mathbb{D}) = H^i(C_{\varphi, \gamma}^\bullet(\mathbb{D})).$$

Then $H^i(\mathbb{D})$ are finitely generated A -modules ([25], [24]). In addition, we have a canonical isomorphism

$$(2.3) \quad H^2(\mathcal{R}_A(\chi)) \simeq A$$

given by $[x] \mapsto -\left(1 - \frac{1}{p}\right)^{-1} \log^{-1} \chi(\gamma) \text{res}(xdt)$.

If $\mathbb{D} = \mathbb{D}_{\text{rig}, A}^\dagger(V)$, where V is a p -adic representation of G_p with coefficients in A , there exist canonical (up to the choice of γ) and functorial isomorphisms

$$(2.4) \quad H^i(\mathbb{D}_{\text{rig}, A}^\dagger(V)) \simeq H^i(\mathbb{Q}_p, V), \quad i \in \mathbb{Z}.$$

(see [25], [24]). Note that $H^i(\mathcal{R}_A(\chi)) \simeq H^i(\mathbb{Q}_p, A(\chi))$.

If $\mathbb{D}^*(\chi)$ is the dual (φ, Γ) -module, then the canonical pairing $\mathbb{D} \otimes \mathbb{D}^*(\chi) \rightarrow \mathcal{R}_E(\chi)$ induces a well defined cup-product

$$\cup_{\varphi, \gamma} : C_{\varphi, \gamma}^\bullet(\mathbb{D}) \otimes C_{\varphi, \gamma}^\bullet(\mathbb{D}^*(\chi)) \rightarrow C_{\varphi, \gamma}^\bullet(\mathcal{R}_E(\chi)).$$

Together with the isomorphism (2.3) this gives the duality for (φ, Γ) -modules

$$C_{\varphi, \gamma}^\bullet(\mathbb{D}) \simeq \mathbf{R}\text{Hom}_A(C_{\varphi, \gamma}^\bullet(\mathbb{D}^*(\chi)), A).$$

For our construction of p -adic heights we need the derived version of the isomorphism (2.4) proved in [6]. Namely, let $\widetilde{\mathbb{B}}_{\text{rig}}^{\dagger, r}$ denote the ring of p -adic periods constructed by Berger in [8]. Set $\widetilde{\mathbb{B}}_{\text{rig}, A}^{\dagger, r} = \widetilde{\mathbb{B}}_{\text{rig}}^{\dagger, r} \widehat{\otimes}_{\mathbb{Q}_p} A$, $\widetilde{\mathbb{B}}_{\text{rig}, A}^\dagger = \bigcup_{r>0} \widetilde{\mathbb{B}}_{\text{rig}, A}^{\dagger, r}$, $V_{\text{rig}}^\dagger = V \otimes_A \widetilde{\mathbb{B}}_{\text{rig}, A}^\dagger$ and consider the complex $C^\bullet(G_p, V_{\text{rig}}^\dagger)$. The exact sequence

$$0 \rightarrow E \rightarrow \widetilde{\mathbb{B}}_{\text{rig}}^\dagger \xrightarrow{\varphi-1} \widetilde{\mathbb{B}}_{\text{rig}}^\dagger \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow C^\bullet(G_p, V) \rightarrow C^\bullet(G_p, V_{\text{rig}}^\dagger) \xrightarrow{\varphi-1} C^\bullet(G_p, V_{\text{rig}}^\dagger) \rightarrow 0.$$

Consider the total complex

$$K_p^\bullet(V) = \text{Tot} \left(C^\bullet(G_p, V_{\text{rig}}^\dagger) \xrightarrow{\varphi-1} C^\bullet(G_p, V_{\text{rig}}^\dagger) \right).$$

The canonical map $V \rightarrow V_{\text{rig}}^\dagger$ induces a morphism

$$\xi : C^\bullet(G_p, V) \rightarrow K_p^\bullet(V).$$

given by

$$\xi(x) = (0, x) \in C^{m-1}(G_p, V_{\text{rig}}^\dagger) \oplus C^m(G_p, V_{\text{rig}}^\dagger), \quad x \in C^m(G_p, V).$$

On the other hand, consider the complex

$$C_\gamma^\bullet(\mathbb{D}_{\text{rig}}^\dagger(V)) : \mathbb{D}_{\text{rig}}^\dagger(V)^\Delta \xrightarrow{\gamma-1} \mathbb{D}_{\text{rig}}^\dagger(V)^\Delta,$$

where the terms are placed in degrees 0 and 1. We have a morphism of complexes

$$\alpha : C_\gamma^\bullet(\mathbb{D}_{\text{rig}}^\dagger(V)) \rightarrow C^\bullet(G_p, V_{\text{rig}}^\dagger)$$

defined by

$$\begin{aligned} \alpha(x) &= x, & x &\in C_\gamma^0(\mathbb{D}_{\text{rig}}^\dagger(V)) \\ \alpha(x)(g) &= \frac{g-1}{\gamma-1}(x) & x &\in C_\gamma^1(\mathbb{D}_{\text{rig}}^\dagger(V)), \quad g \in G_p. \end{aligned}$$

Since $C_{\varphi, \gamma}^\bullet(\mathbb{D}_{\text{rig}}^\dagger(V)) = \text{Tot} \left(C_\gamma^\bullet(\mathbb{D}_{\text{rig}}^\dagger(V)) \xrightarrow{\varphi-1} C_\gamma^\bullet(\mathbb{D}_{\text{rig}}^\dagger(V)) \right)$, this morphism induces a morphism (which we denote again by α):

$$\alpha : C_{\varphi, \gamma}^\bullet(\mathbb{D}_{\text{rig}}^\dagger(V)) \rightarrow K_p^\bullet(V).$$

Proposition 2.2. *The maps α and ξ are quasi-isomorphisms and we have a diagram*

$$\begin{array}{ccc} C^\bullet(G_p, V) & \xrightarrow[\simeq]{\xi} & K_p^\bullet(V), \\ & \alpha \uparrow \simeq & \\ & C_{\varphi, \gamma}^\bullet(\mathbb{D}_{\text{rig}}^\dagger(V)) & \end{array}$$

Proof. This is [6], Proposition 2.4.2. □

2.4. Iwasawa cohomology. For each $n \geq 1$ we set $K_n = \mathbb{Q}_p(\zeta_{p^{n+1}})^\Delta$. Thus K_n/\mathbb{Q}_p is the cyclotomic extension of degree p^n . Let $\Lambda = O_E[[\Gamma_0]]$ denote the Iwasawa algebra over O_E . Then the choice of a generator $\gamma \in \Gamma_0$ fixes an isomorphism $\Lambda \simeq O_E[[X]]$ such that $\gamma \mapsto 1 + X$. We denote by \mathcal{H} the algebra of power series $f(X) = \sum_{i=0}^{\infty} a_i X^i$ with coefficients in E that converge on the p -adic open unit disk. If A is an affinoid E -algebra, we set $\Lambda_A = \Lambda \widehat{\otimes}_{O_E} A$ and $\mathcal{H}_A = \mathcal{H} \widehat{\otimes}_E A$. The map $\iota(\tau) = \tau^{-1}$, $\tau \in \Gamma_0$ defines involutions $\iota : \Lambda_A \rightarrow \Lambda_A$ and $\mathcal{H}_A \rightarrow \mathcal{H}_A$.

Let V be a p -adic representation of G_p with coefficients in an affinoid algebra A . Fix a unit ball A° of A . Then $V = T \otimes_{O_E} E$ for some finitely generated projective A° -submodule T of V stable under G_p . We write $V \otimes_A \Lambda_A^\iota$ (respectively $V \otimes_A \mathcal{H}_A^\iota$) for

$V \otimes_A \Lambda_A$ (respectively $V \widehat{\otimes}_A \mathcal{H}_A$) equipped with the diagonal action of $G_{\mathbb{Q},S}$ and the Λ_A -module (respectively \mathcal{H}_A -module) structure given by

$$\tau(v \otimes \lambda) = a \otimes \tau^{-1}\lambda, \quad \tau \in \Gamma_0, \quad v \in V, \quad \lambda \in \Lambda_A \text{ (resp. } \lambda \in \mathcal{H}_A \text{)}.$$

Consider the complexes $C^\bullet(G_p, V \otimes_A \Lambda_A^\iota)$ and $C^\bullet(G_p, V \otimes_A \mathcal{H}_A^\iota)$ of continuous cochains with coefficients in $V \otimes_A \Lambda_A^\iota$ and $V \otimes_A \mathcal{H}_A^\iota$ respectively. Then

$$H^i(C^\bullet(G_p, V \otimes_A \Lambda_A^\iota)) \simeq H_{\text{Iw}}^i(\mathbb{Q}_p, V),$$

where

$$H_{\text{Iw}}^i(\mathbb{Q}_p, V) = \varprojlim_n H^i(G_{K_n}, T) \otimes_{O_E} E$$

and

$$H^i(\mathbb{Q}_p, V \otimes_A \mathcal{H}_A^\iota) \simeq H_{\text{Iw}}^i(\mathbb{Q}_p, V) \otimes_{\Lambda_A} \mathcal{H}_A$$

(see [35] Theorem 1.6). To simplify notation, we set $\overline{V} = V \otimes_A \mathcal{H}_A^\iota$.

For any (φ, Γ) -module \mathbb{D} over \mathcal{R}_A define $\overline{\mathbb{D}} = \mathbb{D} \otimes_{\mathcal{R}_A} \mathbb{D}_{\text{rig}, A}^\dagger(\mathcal{H}_A^\iota)$ (see [24] and [36] for more detail). The Iwasawa cohomology $H_{\text{Iw}}^\bullet(\mathbb{D})$ of \mathbb{D} is defined to be the cohomology of the complex

$$C_{\text{Iw}}^\bullet(\mathbb{D}) = C_{\varphi, \gamma}^\bullet(\overline{\mathbb{D}}).$$

Note that this complex is quasi-isomorphic to the complex

$$\left[\mathbb{D}^\Delta \xrightarrow{\psi-1} \mathbb{D}^\Delta \right],$$

where ψ is the left inverse to φ and the terms are placed in degrees 1 and 2 ([24], Theorem 4.4.8). We have natural quasi-isomorphisms

$$\begin{aligned} C_{\text{Iw}}^\bullet(\mathbb{D}) \otimes_{\mathcal{H}_A}^\mathbb{L} A &\simeq C_{\varphi, \gamma}^\bullet(\mathbb{D}), \\ C^\bullet(G_p, \overline{V}) &\simeq C_{\text{Iw}}^\bullet(\mathbb{D}_{\text{rig}, A}^\dagger(V)). \end{aligned}$$

In particular,

$$H_{\text{Iw}}^*(\mathbb{Q}_p, V) \otimes_{\Lambda_A} \mathcal{H}_A \simeq H_{\text{Iw}}^*(\mathbb{D}_{\text{rig}, A}^\dagger(V)).$$

2.5. The Bloch–Kato Selmer group. In this section we assume that $A = E$. As usual, the first cohomology group $H^1(\mathbb{D})$ classifies extensions of the trivial (φ, Γ) -module \mathcal{R}_E by \mathbb{D} :

$$0 \rightarrow \mathbb{D} \rightarrow \mathbb{D}_x \rightarrow \mathcal{R}_E \rightarrow 0.$$

We denote by $H_f^1(\mathbb{D})$ the subgroup of crystalline extensions, namely

$$H_f^1(\mathbb{D}) = \{[x] \in H^1(\mathbb{D}) \mid \dim_E \mathcal{D}_{\text{cris}}(\mathbb{D}_x) = \dim_E \mathcal{D}_{\text{cris}}(\mathbb{D}) + 1\}.$$

Note that this definition agrees with H_f^1 of Bloch and Kato [3], [28].

Let $\delta : \mathbb{Q}_p^* \rightarrow E^*$ be a continuous character. The (φ, Γ) -module $\mathcal{R}_E(\delta)$ is semistable (and therefore crystalline) if and only if $\delta|_{\mathbb{Z}_p^*}(u) = u^m$ for some $m \in \mathbb{Z}$. The following proposition summarizes main information about the cohomology of such modules.

Proposition 2.3. *Let $\delta : \mathbb{Q}_p^* \rightarrow E^*$ be a continuous character such that $\delta|_{\mathbb{Z}_p^*}(u) = u^m$ for some $m \in \mathbb{Z}$.*

- i) *If $m \leq 0$ and $\delta \neq x^m$, then $\dim_E H^1(\mathcal{R}_E(\delta)) = 1$ and $H_f^1(\mathcal{R}_E(\delta)) = 0$.*
- ii) *If $m \leq 0$ and $\delta = x^m$, then the map*

$$\begin{aligned} i_\delta : \mathcal{D}_{\text{cris}}(\mathcal{R}_E(\delta)) \times \mathcal{D}_{\text{cris}}(\mathcal{R}_E(\delta)) &\rightarrow H^1(\mathcal{R}_E(\delta)), \\ i_\delta(x, y) &= \text{cl}(t^{-m}x, t^{-m}y), \quad t = \log(1 + \pi) \end{aligned}$$

is an isomorphism. Let $i_{\delta,f}$ and $i_{\delta,c}$ denote the restriction of i_δ on the first and second summand respectively. Then $\text{im}(i_{\delta,f}) = H_f^1(\mathcal{R}_E(\delta))$ and we obtain the decomposition

$$(2.5) \quad H^1(\mathcal{R}_E(\delta)) \stackrel{(\text{pr}_f, \text{pr}_c)}{=} H_f^1(\mathcal{R}_E(\delta)) \oplus H_c^1(\mathcal{R}_E(\delta)),$$

where $H_c^1(\mathcal{R}_E(\delta)) = \text{im}(i_{\delta,c})$.

iii) If $m \geq 1$ and $\delta(x) \neq |x|x^m$, then $H_f^1(\mathcal{R}_E(\delta)) = H^1(\mathcal{R}_E(\delta))$ est un E -espace vectoriel de dimension 1.

iv) If $\delta(x) = |x|x^m$, then $\chi\delta^{-1}(x) = x^{1-m}$. Consider local duality

$$\cup : H^1(\mathcal{R}_E(\delta)) \times H^1(\mathcal{R}_E(\chi\delta^{-1})) \rightarrow E$$

and denote by $[\ , \] : \mathcal{D}_{\text{cris}}(\mathcal{R}_E(\delta)) \times \mathcal{D}_{\text{cris}}(\mathcal{R}_E(\chi\delta^{-1})) \rightarrow E$ the canonical pairing. Then the map

$$i_\delta : \mathcal{D}_{\text{cris}}(\mathcal{R}_E(\delta)) \times \mathcal{D}_{\text{cris}}(\mathcal{R}_E(\delta)) \rightarrow H^1(\mathcal{R}_E(\delta))$$

defined by

$$i_\delta(x, y) \cup i_{\chi\delta^{-1}}(\alpha, \beta) = [y, \alpha] - [x, \beta]$$

is an isomorphism. Let $i_{\delta,f}$ and $i_{\delta,c}$ denote the restriction of i_δ on the first and second summand respectively. Then $\text{im}(i_{\delta,f}) = H_f^1(\mathcal{R}_E(\delta))$ and again we have the decomposition (2.5) with $H_c^1(\mathcal{R}_E(\delta)) = \text{im}(i_{\delta,c})$.

Proof. See Proposition 1.5.3, Proposition 1.5.4 and Theorem 1.5.7 of [3]. \square

2.6. Selmer complexes. In this subsection we review the construction of Selmer complexes for non-ordinary Galois representations following [35] and [6]. Fix a finite set of primes S such that $p \in S$. Let $G_{\mathbb{Q},S}$ denote the Galois group of the maximal algebraic extension of \mathbb{Q} unramified outside S and ∞ . For each prime ℓ we fix a decomposition group at ℓ which we identify with $G_\ell = \text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$, and denote by I_ℓ the inertia subgroup of G_ℓ . If G is a topological group and M a continuous G -module, we will write $C^\bullet(G, M)$ for the complex of continuous cochains of G with coefficients in M .

Let V be a p -adic representation of $G_{\mathbb{Q},S}$ with coefficients in an affinoid algebra A . We denote by $\mathbb{D}_{\text{rig},A}^\dagger(V)$ the (φ, Γ) -module associated to the restriction of V on the decomposition group at p . Let \mathbb{D} be a (φ, Γ) -submodule of $\mathbb{D}_{\text{rig},A}^\dagger(V)$ such that \mathbb{D} is an \mathcal{R}_A -direct summand of $\mathbb{D}_{\text{rig},A}^\dagger(V)$. Set $U_p^+(V, \mathbb{D}) = C_{\varphi, \gamma}^\bullet(\mathbb{D})$. Composing the quasi-isomorphism α of Proposition 2.2 with the canonical morphism $U_p^+(V, \mathbb{D}) \rightarrow C_{\varphi, \gamma}^\bullet(\mathbb{D}_{\text{rig},A}^\dagger(V))$, we obtain a map

$$i_p^+ : U_p^+(V, \mathbb{D}) \rightarrow K_p^\bullet(V),$$

which we will consider as a local condition at p . For each $\ell \in S \setminus \{p\}$ we set

$$U_\ell^+(V) = \left[V^{I_\ell} \xrightarrow{\text{Fr}_\ell - 1} V^{I_\ell} \right],$$

where Fr_ℓ denotes the geometric Frobenius and the terms are placed in degrees 0 and 1. Define

$$i_\ell^+ : U_\ell^+(V) \rightarrow C^\bullet(G_{\mathbb{Q},S}, V)$$

by

$$\begin{aligned} i_\ell^+(x) &= x && \text{in degree 0,} \\ (i_\ell^+(x))(\text{Fr}_\ell) &= x && \text{in degree 1.} \end{aligned}$$

Set

$$K_S^\bullet(V) = K_p^\bullet(V) \oplus \left(\bigoplus_{\ell \in S \setminus \{p\}} C^\bullet(G_\ell, V) \right)$$

and $U_S^+(V, \mathbb{D}) = U_p^+(V, \mathbb{D}) \oplus \left(\bigoplus_{\ell \in S \setminus \{p\}} U_\ell^+(V) \right)$. To uniformize notation, we will often write $K_\ell^\bullet(V)$ for $C^\bullet(G_\ell, V)$ if $\ell \neq p$. We have a diagram

$$\begin{array}{ccc} C^\bullet(G_{\mathbb{Q}, S}, V) & \xrightarrow{\text{res}_S} & K_S^\bullet(V) \\ & & \uparrow i_S^+ \\ & & U_S^+(V, \mathbb{D}), \end{array}$$

where $i_S^+ = (i_\ell^+)_{\ell \in S}$ and res_S denotes the localization map.

Definition 2.4. The Selmer complex associated to these data is defined as

$$S^\bullet(V, \mathbb{D}) = \text{cone} \left[C^\bullet(G_{\mathbb{Q}, S}, V) \oplus U_S^+(V) \xrightarrow{\text{res}_S - i_S^+} K_S^\bullet(V) \right] [-1].$$

Each element $x_f \in S^i(V, \mathbb{D})$ may be written as a triple

$$x_f = (x, (x_\ell^+)_{\ell \in S}, (\lambda_\ell)_{\ell \in S}),$$

where $x \in C^i(G_{\mathbb{Q}, S}, V)$, $(x_\ell^+)_{\ell \in S} \in U_S^+(V)$ and $(\lambda_\ell)_{\ell \in S} \in K_S^{i-1}(V)$, and x_f is a cocycle if and only if

$$i_S((x_\ell^+)_{\ell \in S}) = \text{res}_S(x) + d((\lambda_\ell)_{\ell \in S}).$$

To simplify notation, we will often write $x_f = (x, (x_\ell^+), (\lambda_\ell))$ in place of $x_f = (x, (x_\ell^+)_{\ell \in S}, (\lambda_\ell)_{\ell \in S})$.

Definition 2.5. We denote by $\mathbf{R}\Gamma(V, \mathbb{D})$ the class of $S^\bullet(V, \mathbb{D})$ in the derived category of A -modules and define

$$H^i(V, \mathbb{D}) := \mathbf{R}^i\Gamma(V, \mathbb{D}).$$

For each cocycle $x \in S^i(V, \mathbb{D})$ we write $[x]$ for the class of x in $H^i(V, \mathbb{D})$.

Definition 2.6. For each $\ell \in S$ we define a complex

$$\tilde{U}_\ell(V, \mathbb{D}) := \text{cone} \left(U_\ell^+(V, \mathbb{D}) \xrightarrow{-i_\ell^+} K_\ell^\bullet(V) \right).$$

and set $\tilde{U}_S(V, \mathbb{D}) := \bigoplus_{\ell \in S} \tilde{U}_\ell(V, \mathbb{D})$.

We have the following tautological exact triangles in the derived category

$$(2.6) \quad \tilde{U}_S(V, \mathbb{D})[-1] \longrightarrow \mathbf{R}\Gamma(V, \mathbb{D}) \longrightarrow \mathbf{R}\Gamma(G_{\mathbb{Q}, S}, V) \xrightarrow{\widetilde{\text{res}}_S} \tilde{U}_S(V, \mathbb{D}),$$

and for each $\ell \in S$,

$$(2.7) \quad U_\ell^+(V, \mathbb{D}) \longrightarrow K_\ell^\bullet(V) \longrightarrow \tilde{U}_\ell(V, \mathbb{D}) \longrightarrow U_\ell^+(V, \mathbb{D})[1].$$

2.7. Duality for Selmer complexes. We review the duality theory for Selmer complexes. As usual, we denote by $\tau_{\geq m}$ the truncation map. Define

$$Z^\bullet = \text{cone} \left(\tau_{\geq 2} C^\bullet(G_{\mathbb{Q}, S}, A(1)) \xrightarrow{\text{res}_S} \tau_{\geq 2} K_S^\bullet(A(1)) \right) [-1].$$

The computation of the Brauer group in Class Field Theory yields an exact sequence

$$0 \rightarrow H_S^2(A(1)) \rightarrow \bigoplus_{l \in S} H^2(\mathbb{Q}_l, A(1)) \rightarrow A \rightarrow 0,$$

which allows to construct a canonical, up to homotopy, quasi-isomorphism

$$r_S : Z^\bullet \simeq A[-3]$$

(see [31], Section 5.4.1). The formula

$$(a_{i-1}, a_i) \cup_{K, p} (b_{j-1}, b_j) = (a_i \cup b_{j-1} + (-1)^j a_{i-1} \cup \varphi(b_j), a_i \cup b_j),$$

where \cup denotes the cup-product of continuous Galois cochains, defines a morphism of complexes

$$\cup_{K, p} : K_p^\bullet(V) \otimes K_p^\bullet(V^*(1)) \rightarrow K_p^\bullet(A(1))$$

(see [6], Proposition 1.1.5). To simplify notation, we will write \cup instead $\cup_{K, p}$.

Let $\mathbb{D}^\perp = \text{Hom}_{\mathcal{R}}(\mathbb{D}_{\text{rig}}^\dagger(V)/\mathbb{D}, \mathcal{R}_A(\chi))$ be the orthogonal complement to \mathbb{D} under the canonical duality $\mathbb{D}_{\text{rig}, A}^\dagger(V) \times \mathbb{D}_{\text{rig}, A}^\dagger(V^*(1)) \rightarrow \mathcal{R}_A(\chi)$. Following Nekovář, we define a pairing

$$\cup_{V, \mathbb{D}} : S^\bullet(V, \mathbb{D}) \otimes_A S^\bullet(V^*(1), \mathbb{D}^\perp) \rightarrow A[-3]$$

as the composition map

$$S^\bullet(V, \mathbb{D}) \otimes_A S^\bullet(V^*(1), \mathbb{D}^\perp) \rightarrow Z^\bullet \xrightarrow{r_S} A[-3],$$

where the first arrow is induced by the cup-product

$$(2.8) \quad (x, (x_\ell^+), (\lambda_\ell)) \cup (y, (y_\ell^+), (\mu_\ell)) = (x \cup y, (\lambda_\ell \cup i_S^+(y_\ell^+) + (-1)^{\deg(x)} \text{res}_S(x) \cup \mu_\ell)).$$

Therefore we have a morphism in the derived category of A -modules

$$(2.9) \quad \mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A^{\mathbb{L}} \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) \rightarrow A[-3].$$

Proposition 2.7. *We have a commutative diagram*

$$\begin{array}{ccc} \mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A^{\mathbb{L}} \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) & \xrightarrow{\cup_{V, \mathbb{D}}} & A[-3] \\ \downarrow s_{12} & & \downarrow = \\ \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) \otimes_A^{\mathbb{L}} \mathbf{R}\Gamma(V, \mathbb{D}) & \xrightarrow{\cup_{V^*(1), \mathbb{D}^\perp}} & A[-3] \end{array}$$

where $s_{12}(x \otimes y) = (-1)^{\deg(x) \deg(y)} y \otimes x$.

Proof. This is [6], Theorem 3.1.8. □

If $A = E$ is a finite extension of \mathbb{Q}_p , this pairing induces a canonical duality

$$\text{Hom}_E(\mathbf{R}\Gamma(V, \mathbb{D}), E) \simeq \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp),$$

but this is not true in general.

We also review here the Iwasawa theoretic analog of $\mathbf{R}\Gamma(V, \mathbb{D})$. Recall that $\overline{V} = V \otimes_A \mathcal{H}_A^\iota$ and $\overline{\mathbb{D}} = \mathbb{D} \otimes_{\mathcal{R}_A} \mathbb{D}_{\text{rig}, A}^\dagger(\mathcal{H}_A^\iota)$. With the previous notations define

$$S_{\text{Iw}}^\bullet(V, \mathbb{D}) = S^\bullet(\overline{V}, \overline{\mathbb{D}}).$$

We will write $\mathbf{R}\Gamma_{\text{Iw}}(V, \mathbb{D})$ for the class of $S_{\text{Iw}}^\bullet(V, \mathbb{D})$ in the derived category of \mathcal{H}_A -modules. Note that the following version of the control theorem holds true:

$$\mathbf{R}\Gamma_{\text{Iw}}(V, \mathbb{D}) \otimes_{\mathcal{H}}^{\mathbb{L}} E = \mathbf{R}\Gamma(V, \mathbb{D}).$$

3. p -ADIC HEIGHT PAIRINGS

3.1. Construction of p -adic heights. We provide in this section an overview of the construction of p -adic heights for p -adic representations over affinoid algebras following [6]. We keep previous notation and conventions. Let A be an affinoid algebra over E and let V be a p -adic representation of $G_{\mathbb{Q}, S}$ with coefficients in A . We fix a (φ, Γ) -submodule \mathbb{D} of $\mathbb{D}_{\text{rig}, A}^\dagger(V)$ which is a \mathcal{R}_A -module direct summand of $\mathbb{D}_{\text{rig}, A}^\dagger(V)$. Let J_A denote the kernel of the augmentation map $\mathcal{H}_A \rightarrow A$. Note that $J_A = (X)$ and $J_A/J_A^2 \simeq A$ as A -modules. Since Selmer complexes commute with base change (see [35], Theorem 1.12), the exact sequence

$$0 \rightarrow J_A/J_A^2 \rightarrow \mathcal{H}_A/J_A^2 \rightarrow A \rightarrow 0$$

gives rise to a distinguished triangle

$$\mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A J_A/J_A^2 \rightarrow \mathbf{R}\Gamma_{\text{Iw}}(V, \mathbb{D}) \otimes_{\mathcal{H}}^{\mathbb{L}} \mathcal{H}/(X^2) \rightarrow \mathbf{R}\Gamma(V, \mathbb{D}) \xrightarrow{\beta_{V, \mathbb{D}}^{\text{cyc}}} \mathbf{R}\Gamma(V, \mathbb{D})[1] \otimes_A J_A/J_A^2.$$

Definition 3.1. The p -adic height pairing associated to the data (V, \mathbb{D}) is defined as the morphism

$$h_{V, \mathbb{D}} : \mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A^{\mathbb{L}} \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) \xrightarrow{\beta_{V, \mathbb{D}}^{\text{cyc}} \otimes \text{id}} (\mathbf{R}\Gamma(V, \mathbb{D})[1] \otimes J_A/J_A^2) \otimes_E^{\mathbb{L}} \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) \xrightarrow{\cup_{V, \mathbb{D}}} J_A/J_A^2[-2]$$

The pairing $h_{V, \mathbb{D}}$ induces a pairing on cohomology groups

$$h_{V, \mathbb{D}} : H^1(V, \mathbb{D}) \times H^1(V^*(1), \mathbb{D}^\perp) \rightarrow J_A/J_A^2.$$

Proposition 3.2. *The following diagram*

$$\begin{array}{ccc} \mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A^{\mathbb{L}} \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) & \xrightarrow{h_{V, \mathbb{D}}} & J_A/J_A^2[-2] \\ \downarrow s_{12} & & \downarrow = \\ \mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) \otimes_A^{\mathbb{L}} \mathbf{R}\Gamma(V, \mathbb{D}) & \xrightarrow{h_{V^*(1), \mathbb{D}^\perp}} & J_A/J_A^2[-2], \end{array}$$

where $s_{12}(a \otimes b) = (-1)^{\deg(a) \deg(b)} b \otimes a$, commutes.

In particular,

$$h_{V, \mathbb{D}}^{i, j}(x, y) = (-1)^{ij} h_{V^*(1), \mathbb{D}^\perp}^{j, i}(y, x).$$

Proof. This is Theorem I of [6]. □

Till the end of this subsection we assume that $A = E$ is a finite extension of \mathbb{Q}_p and that the restriction of V to the decomposition group at p is semistable.

Definition 3.3. Assume that V is semistable at p . We say that a (φ, N) -submodule D of $\mathbb{D}_{\text{st}}(V)$ is a splitting submodule if

$$(3.1) \quad \mathbb{D}_{\text{st}}(V) = D \oplus \text{Fil}^0 \mathbb{D}_{\text{st}}(V)$$

as E -vector spaces.

Let D be a splitting submodule of $\mathbb{D}_{\text{st}}(V)$. By [10], D corresponds to a unique (φ, Γ) -submodule \mathbb{D} of $\mathbb{D}_{\text{rig}}^\dagger(V)$ such that $\mathcal{D}_{\text{st}}(\mathbb{D}) = D$. To simplify notation, we write $\mathbf{R}\Gamma(V, D)$ and $h_{V,D}$ for $\mathbf{R}\Gamma(V, \mathbb{D})$ and $h_{V,\mathbb{D}}$ respectively.

Proposition 3.4. *Let D be a splitting submodule of V . Assume that the following conditions hold true:*

$$\text{a) } \mathbb{D}_{\text{cris}}(V)^{\varphi=1} = \mathbb{D}_{\text{cris}}(V^*(1))^{\varphi=1} = 0;$$

$$\text{b) } H^0(\mathbb{D}_{\text{rig}}^\dagger(V)/\mathbb{D}) = H^0(\mathbb{D}^*(\chi)) = 0, \text{ where } \mathbb{D}^* = \text{Hom}_{\mathcal{R}}(\mathbb{D}, \mathcal{R}).$$

Then we have $\mathbf{R}^1\Gamma(V, D) = H_f^1(\mathbb{Q}, V)$, $\mathbf{R}^1\Gamma(V^(1), D^\perp) = H_f^1(\mathbb{Q}, V^*(1))$, and*

$$h_{V,D} : H_f^1(\mathbb{Q}, V) \times H_f^1(\mathbb{Q}, V^*(1)) \rightarrow E$$

coincides with the p -adic height pairing constructed by Nekovář in [30] ².

Proof. See [6], Theorem III. □

Even when the splitting submodule D does not satisfy the condition b) of Proposition 3.4, we can still relate $\mathbf{R}^1\Gamma(V, D)$ to the Bloch-Kato Selmer group if the restriction of V on the decomposition group at p satisfies some natural conditions. Conjecturally, such a situation appears if the associated p -adic L -function has an extra-zero. We refer to [6, Section 7], for a systematic study of p -adic heights in this setting. In Section 4 we review this theory in the particular case of elliptic modular forms.

3.2. Cassels–Tate pairings. Nekovář's construction of abstract Cassels-Tate pairings generalizes directly to our case. Let A be an affinoid algebra. We assume that A is an integral domain and denote by $\text{Fr}(A)$ its field of fractions. Let V be a p -adic representation of $G_{\mathbb{Q},S}$ with coefficients in A and let \mathbb{D} be a (φ, Γ) -submodule of $\mathbb{D}_{\text{rig},A}^\dagger(V)$ such that \mathbb{D} is a \mathcal{R}_A -module direct summand of $\mathbb{D}_{\text{rig}}^\dagger(V)$. Consider the complex of flat modules

$$C^\bullet = \left[A \xrightarrow{-\text{id}} \text{Fr}(A) \right]$$

placed in degrees 0 and 1. Let X^\bullet be a complex of A -modules. The Tor spectral sequence for $X^\bullet \otimes_A C^\bullet$ degenerates into exact sequences

$$(3.2) \quad 0 \rightarrow H^{i-1}(X^\bullet) \otimes_A (\text{Fr}(A)/A) \rightarrow H^i(X^\bullet \otimes_A C^\bullet) \rightarrow H^i(X^\bullet)_{\text{tor}} \rightarrow 0.$$

Applying the functor $\otimes_A C^\bullet$ to the pairing (2.9) and passing to cohomology groups, we get pairings

$$H^i(\mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A C^\bullet) \otimes_A H^j(\mathbf{R}\Gamma(V^*(1), \mathbb{D}^\perp) \otimes_A C^\bullet) \rightarrow \text{Fr}(A)/A, \quad i + j = 4.$$

Since $\text{Fr}(A)/A$ is A -divisible, it follows from (3.2) that this pairing factors through

$$(3.3) \quad \text{CT}_{V,\mathbb{D}}^{i,j} : H^i(V, \mathbb{D})_{\text{tor}} \otimes_A H^j(V^*(1), \mathbb{D}^\perp)_{\text{tor}} \rightarrow \text{Fr}(A)/A.$$

Definition 3.5. We will call $\text{CT}_{V,\mathbb{D}}^{i,j}$ generalized Cassels-Tate pairings for (V, \mathbb{D}) .

Proposition 3.6. *The pairings $\text{CT}_{V,\mathbb{D}}^{i,j}$ satisfy*

$$\text{CT}_{V,\mathbb{D}}^{i,j}(x, y) = (-1)^{ij} \text{CT}_{V,\mathbb{D}}^{j,i}(y, x), \quad i + j = 4.$$

Proof. This follows from Proposition 2.7 and [31], formula (2.10.14.1). □

²The p -adic height pairing constructed by Nekovář depends on the choice of splitting of the Hodge filtration. One should take here the splitting defined by the decomposition (3.1).

3.3. The pairing $h_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}}^{\text{wt}}$. We maintain the assumptions of Section 3.2. Assume in addition that A is a principal ideal domain. Fix an E -point of $U = \text{Spm}(A)$ which we will identify with a morphism $\psi : A \rightarrow E$ and set $\mathfrak{p} = \ker(\psi)$. Let $V_{\mathfrak{p}} = V \otimes_{A, \psi} E$ and $\mathbb{D}_{\mathfrak{p}} = \mathbb{D} \otimes_{A, \psi} E$. The exact sequence

$$0 \rightarrow \mathfrak{p} \rightarrow A \rightarrow E \rightarrow 0$$

induces a distinguished triangle

$$\mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A \mathfrak{p} \rightarrow \mathbf{R}\Gamma(V, \mathbb{D}) \rightarrow \mathbf{R}\Gamma(V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}) \rightarrow \mathbf{R}\Gamma(V, \mathbb{D})[1] \otimes_A \mathfrak{p}.$$

Since \mathfrak{p} is free over A , we have $H^i(\mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A \mathfrak{p}) = \mathbf{R}\Gamma(V, \mathbb{D}) \otimes_A \mathfrak{p}$, and this distinguished triangle induces an exact sequence

$$\dots \rightarrow H^1(V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}) \xrightarrow{\mu_{V, \mathbb{D}}^{\text{wt}}} H^2(V, \mathbb{D}) \otimes_A \mathfrak{p} \rightarrow H^2(V, \mathbb{D}) \rightarrow \dots$$

Since $\ker(H^2(V, \mathbb{D}) \otimes_A \mathfrak{p} \rightarrow H^2(V, \mathbb{D})) = H^2(V, \mathbb{D})_{\mathfrak{p}\text{-tor}} \otimes_A \mathfrak{p}$, we can compose $\mu_{V, \mathbb{D}}^{\text{wt}}$ with the pairing (3.5).

Definition 3.7. The weight height pairing is defined to be the E -bilinear map

$$(3.4) \quad h_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}}^{\text{wt}} : H^1(V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}) \otimes_E H^1(V_{\mathfrak{p}}^*(1), \mathbb{D}_{\mathfrak{p}}^{\perp}) \xrightarrow{\mu_{V, \mathbb{D}}^{\text{wt}} \otimes \mu_{V^*(1), \mathbb{D}^{\perp}}^{\text{wt}}} (H^2(V, \mathbb{D})_{\mathfrak{p}\text{-tor}} \otimes_A \mathfrak{p}) \otimes_A (H^2(V^*(1), \mathbb{D}^{\perp})_{\mathfrak{p}\text{-tor}} \otimes_A \mathfrak{p}) \xrightarrow{\text{CT}_{V, \mathbb{D}}^{2,2}} \mathfrak{p}^{-1} A/A \otimes_A \mathfrak{p}^2 \simeq \mathfrak{p}/\mathfrak{p}^2.$$

We remark that the pairing $h_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}}^{\text{wt}}$ is symmetric by Proposition 3.6, namely

$$h_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}}^{\text{wt}}(x, y) = h_{V_{\mathfrak{p}}^*(1), \mathbb{D}_{\mathfrak{p}}^{\perp}}^{\text{wt}}(y, x).$$

Tensoring the exact sequence

$$0 \rightarrow \mathfrak{p}/\mathfrak{p}^2 \rightarrow A/\mathfrak{p}^2 \rightarrow E \rightarrow 0$$

with $\mathbf{R}\Gamma(V, \mathbb{D})$, we get the coboundary map

$$\beta_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}}^{\text{wt}} : H^1(V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}) \rightarrow H^2(V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}) \otimes \mathfrak{p}/\mathfrak{p}^2.$$

Proposition 3.8. *We have*

$$h_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}}^{\text{wt}} = \beta_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}}^{\text{wt}}(x) \cup_{V_{\mathfrak{p}}, \mathbb{D}_{\mathfrak{p}}} y.$$

Proof. The proof of this proposition is purely formal and repeats verbatim the proof of Proposition 0.17 of [41] (where only the ordinary case is considered.) \square

4. SELMER COMPLEXES AND p -ADIC HEIGHTS FOR MODULAR FORMS

4.1. Selmer complexes for modular forms. In this Section, we consider p -adic representations arising from elliptic modular forms. Fix an integer $N \geq 1$ such that $p \nmid N$ and set $S = \{\text{primes } \ell \mid N\} \cup \{p\}$. Let $f = \sum_{n=1}^{\infty} a_n q^n$ be an elliptic newform of even weight k for $\Gamma_0(Np)$. We denote by W_f the p -adic representation associated to f by Deligne and set $V_f = W_f(k/2)$. Thus V_f is a two-dimensional representation with coefficients in a finite extension E of \mathbb{Q}_p which is semistable at p . The canonical pairing $W_f \times W_f \rightarrow E(1-k)$ induces an isomorphism

$$(4.1) \quad j : V_f \simeq V_f^*(1).$$

Since the pairing $V_f \times V_f \rightarrow E(1)$ is skew-symmetric, we have an anticommutative diagram

$$(4.2) \quad \begin{array}{ccc} V_f \otimes V_f & \xrightarrow{\text{id} \otimes j} & V_f \otimes V_f^*(1) \\ \downarrow j \otimes \text{id} & & \downarrow \\ V_f^*(1) \otimes V_f & \longrightarrow & E. \end{array}$$

Let $\mathbb{D}_{\text{st}}(V_f)$ denote Fontaine's semistable module associated to V_f . Then $\mathbb{D}_{\text{st}}(V_f)$ is a two-dimensional E -vector space equipped with a decreasing two-step filtration, a Frobenius operator φ , and a monodromy N given by

$$\begin{aligned} \mathbb{D}_{\text{st}}(V_f) &= Ee_\alpha + Ee_\beta, \text{ where } \varphi(e_\alpha) = \alpha e_\alpha, \varphi(e_\beta) = \beta e_\beta, \\ N(e_\beta) &= e_\alpha, \text{ and } N(e_\alpha) = 0, \\ \beta &= p\alpha, \text{ and } \alpha = p^{-k/2}a_p, \\ \text{Fil}^i \mathbb{D}_{\text{st}}(V_f) &= \begin{cases} \mathbb{D}_{\text{st}}(V_f), & \text{if } i \leq -k/2, \\ E(e_\beta - \mathcal{L}_{\text{FM}}(f)e_\alpha), & \text{if } -k/2 + 1 \leq i \leq k/2 - 1, \\ 0, & \text{if } i \geq k/2. \end{cases} \end{aligned}$$

The element $\mathcal{L}_{\text{FM}}(f) \in E$ that appear in the description of the filtration $(\text{Fil}^i \mathbb{D}_{\text{st}}(V))_{i \in \mathbb{Z}}$ is called the Fontaine–Mazur \mathcal{L} -invariant.

We remark that $D = Ee_\alpha$ is the unique non-trivial (φ, N) -submodule of $\mathbb{D}_{\text{st}}(V_f)$. Let \mathbb{D}_f denote the associated (φ, Γ) -submodule of $\mathbb{D}_{\text{rig}}^\dagger(V_f)$. We have a tautological exact sequence

$$(4.3) \quad 0 \rightarrow \mathbb{D}_f \xrightarrow{g} \mathbb{D}_{\text{rig}}^\dagger(V_f) \xrightarrow{h} \widetilde{\mathbb{D}}_f \rightarrow 0.$$

Consider the Selmer complex associated to (V_f, \mathbb{D}_f) . In order to simplify notation, we will set $\widetilde{C}_f^\bullet(V_f) = S^\bullet(V_f, \mathbb{D}_f)$ and $\widetilde{H}_f^i(V_f) = \mathbf{R}^i \Gamma(V_f, \mathbb{D}_f)$. The composition of the p -adic height pairing

$$(4.4) \quad h_{V_f, D} : \widetilde{H}_f^1(V_f) \times \widetilde{H}_f^1(V_f^*(1)) \rightarrow J_E/J_E^2$$

with the isomorphism $\widetilde{H}_f^1(V_f) \simeq \widetilde{H}_f^1(V_f^*(1))$ induced by (4.1) and the isomorphism

$$\begin{aligned} J_E/J_E^2 &\simeq E, \\ \gamma - 1 \pmod{J_E^2} &\longleftrightarrow \log \chi(\gamma) \end{aligned}$$

gives an E -valued pairing

$$\mathfrak{h}_p : \widetilde{H}_f^1(V_f) \times \widetilde{H}_f^1(V_f) \longrightarrow E.$$

From Proposition 3.2 and the anticommutativity of (4.2) it follows that \mathfrak{h}_p is symmetric.

We would like to compare $\widetilde{H}_f^1(V_f)$ with the classical Bloch-Kato Selmer group $H_f^1(\mathbb{Q}_p, V_f)$. It follows from Proposition 3.4 that

$$\widetilde{H}_f^1(V_f) = H_f^1(\mathbb{Q}, V_f), \quad \text{if } a_p \neq p^{k/2-1}.$$

In the remainder of this subsection we assume that $a_p = p^{k/2-1}$. Then $\mathbb{D}_f = \mathcal{R}_E(\delta)$ where $\delta(p) = \alpha p^{k/2}$ and $\delta(u) = u^{k/2}$, $u \in \mathbb{Z}_p^*$. The quotient $\widetilde{\mathbb{D}}_f = \mathbb{D}_{\text{rig}}^\dagger(V)/\mathbb{D}_f$ is a

one-dimensional (φ, Γ) -module which is isomorphic to $\mathcal{R}_E(\tilde{\delta})$ with $\tilde{\delta}(p) = \beta p^{1-k/2}$ and $\tilde{\delta}(u) = u^{1-k/2}$, $u \in \mathbb{Z}_p^*$.

Proposition 4.1. *Assume that $a_p = p^{k/2-1}$. Then*

i) $\mathbb{D}_f \simeq \mathcal{R}_E(|x|x^{k/2})$ and $\tilde{\mathbb{D}}_f \simeq \mathcal{R}_E(x^{1-k/2})$.

ii) *The exact sequence (4.3) induces a long exact sequence*

$$0 \rightarrow H^0(\tilde{\mathbb{D}}_f) \xrightarrow{\partial_0} H^1(\mathbb{D}_f) \xrightarrow{\mathfrak{r}_1} H^1(G_p, V_f) \xrightarrow{\mathfrak{s}_1} H^1(\tilde{\mathbb{D}}_f) \xrightarrow{\partial_1} H^2(\mathbb{D}_f) \rightarrow 0,$$

where we also have $\dim_E H^0(\tilde{\mathbb{D}}_f) = \dim_E H^2(\mathbb{D}_f) = 1$. Moreover,

$$H^1(\mathbb{D}_f) = \text{im}(\partial_0) \oplus H_{\mathfrak{f}}^1(\mathbb{D}_f), \quad \text{im}(\mathfrak{r}_1) = H_{\mathfrak{f}}^1(G_p, V_f), \quad \text{and} \quad H^1(\tilde{\mathbb{D}}_f) = \text{im}(\mathfrak{s}_1) \oplus H_{\mathfrak{f}}^1(\tilde{\mathbb{D}}_f).$$

Proof. The first part is obvious and the second is a particular case of [3, Lemma 2.1.8]. \square

We continue to assume that $a_p = p^{k/2-1}$. Consider the maps

$$\begin{aligned} \partial_c &: H^0(\tilde{\mathbb{D}}_f) \xrightarrow{\partial_0} H^1(\mathbb{D}_f) \xrightarrow{\text{pr}_c} H_c^1(\mathbb{D}_f), \\ \partial_f &: H^0(\tilde{\mathbb{D}}_f) \xrightarrow{\partial_0} H^1(\mathbb{D}_f) \xrightarrow{\text{pr}_f} H_{\mathfrak{f}}^1(\mathbb{D}_f) \end{aligned}$$

and denote by $\rho_c : H^0(\tilde{\mathbb{D}}_f) \rightarrow D$ and $\rho_f : H^0(\tilde{\mathbb{D}}_f) \rightarrow D$ the compositions of these maps with the canonical isomorphisms $H_c^1(\tilde{\mathbb{D}}_f) \simeq D$ and $H_{\mathfrak{f}}^1(\tilde{\mathbb{D}}_f) \simeq D$, respectively. This discussion may be summarized in the following diagram:

$$\begin{array}{ccc} D & \xrightarrow{i_{\delta,c}} & H_c^1(\mathbb{D}_f) \\ \rho_c \uparrow & \nearrow \partial_c & \uparrow \text{pr}_c \\ H^0(\tilde{\mathbb{D}}_f) & \xrightarrow{\partial_0} & H^1(\mathbb{D}_f) \\ \rho_f \downarrow & \searrow \partial_f & \downarrow \text{pr}_f \\ D & \xrightarrow{i_{\delta,f}} & H_{\mathfrak{f}}^1(\mathbb{D}_f) \end{array}$$

Note that from Proposition 4.1 it follows that the maps ∂_c and ρ_c are isomorphisms, and we have a well defined map

$$\partial_c^{-1} \circ \text{pr}_c : H^1(\mathbb{D}_f) \rightarrow H_c^1(\mathbb{D}_f) \rightarrow H^0(\tilde{\mathbb{D}}_f).$$

Proposition 4.2. *Assume that $a_p = p^{k/2-1}$.*

i) *We have an exact sequence*

$$(4.5) \quad 0 \rightarrow H^0(\tilde{\mathbb{D}}_f) \rightarrow \tilde{H}_{\mathfrak{f}}^1(V_f) \rightarrow H_{\mathfrak{f}}^1(\mathbb{Q}, V_f) \rightarrow 0.$$

Moreover, the map $\text{spl} : \tilde{H}_{\mathfrak{f}}^1(V_f) \rightarrow H^0(\tilde{\mathbb{D}}_f)$ given by

$$\text{spl}([x, (x_{\ell})_{\ell \in S}, (\lambda_{\ell})_{\ell \in S}]) = \partial_c^{-1} \circ \text{pr}_c(x_p)$$

defines a canonical splitting of (4.5).

ii) *The composition map $\rho_c^{-1} \circ \rho_f : D \rightarrow D$ coincides with the multiplication by $\mathcal{L}_{\text{FM}}(f)$.*

Proof. The first statement is proved in [6], Proposition 7.1.5. The second statement is proved in [3], p. 1619, formula (32). \square

We fix some notation that is relevant to the constructions above and which will be used in Theorem 4.10 below. We continue to assume that $a_p = p^{k/2-1}$.

Definition 4.3. Let $\Psi_1 := [(-t^{k/2-1}e_{\tilde{\delta}}, 0)]$ and $\Psi_2 := \log \chi(\gamma)[(0, t^{k/2-1}e_{\tilde{\delta}})]$ be two elements of $H^1(\widetilde{\mathbb{D}}_f)$.

We note that $\{\Psi_1, \Psi_2\}$ is a basis of $H^1(\widetilde{\mathbb{D}}_f)$ by Proposition 2.3. Furthermore, Ψ_1 spans the crystalline subspace $H_f^1(\widetilde{\mathbb{D}}_f)$ and Ψ_2 spans the subspace $H_c^1(\widetilde{\mathbb{D}}_f)$. Let $\{\Psi_1^*, \Psi_2^*\}$ denote the skew-dual basis of $H^1(\mathbb{D}_f)$, in the sense that we have

$$\langle \Psi_1, \Psi_1^* \rangle = 1, \quad \langle \Psi_2, \Psi_2^* \rangle = -1, \quad \langle \Psi_1, \Psi_2^* \rangle = \langle \Psi_2, \Psi_1^* \rangle = 0,$$

where

$$\langle \cdot, \cdot \rangle : H^1(\widetilde{\mathbb{D}}_f) \times H^1(\mathbb{D}_f) \rightarrow E$$

denotes the cohomological pairing induced by the canonical pairing $V_f \times V_f \rightarrow E(1)$.

These elements are compared with those defined in [4, Section 1.2.5] in the following way:

$$\Psi_1 = \mathbf{x}_{k/2-1}, \quad \Psi_2 = \mathbf{y}_{k/2-1}, \quad \Psi_1^* = \boldsymbol{\beta}_{k/2}, \quad \text{and} \quad \Psi_2^* = \boldsymbol{\alpha}_{k/2}.$$

Corollary 4.4. A class $[x, (x_\ell^+)_{\ell \in S}, (\lambda_\ell)_{\ell \in S}] \in \widetilde{H}_f^1(V_f)$ is the lift of $[x] \in H_f^1(\mathbb{Q}, V_f)$ under the splitting of Proposition 4.2(i) if and only if $\langle \Psi_1, [x_p^+] \rangle = 0$.

Proof. Note that a class $[x_f] = [x, (x_\ell^+)_{\ell \in S}, (\lambda_\ell)_{\ell \in S}]$ is the lift of a class $[x] \in H_f^1(\mathbb{Q}, V_f)$ under the splitting of Proposition 4.2(i) iff

$$\begin{aligned} [x_f] \in \ker(\text{spl}) &= \left\{ [x_f] = [x, (x_\ell^+)_{\ell \in S}, (\lambda_\ell)_{\ell \in S}] \in \widetilde{H}_f^1(V_f) : \partial_c^{-1} \circ \text{pr}_c([x_p^+]) = 0 \right\} \\ &= \left\{ [x_f] \in \widetilde{H}_f^1(V_f) : \text{pr}_c([x_p^+]) = 0 \right\} \\ &= \left\{ [x_f] \in \widetilde{H}_f^1(V_f) : [x_p^+] \in \text{span}\{\Psi_2^*\} \right\} \\ &= \left\{ [x_f] \in \widetilde{H}_f^1(V_f) : \langle \Psi_1, [x_p^+] \rangle = 0 \right\} \end{aligned}$$

as we have claimed. \square

4.2. p -adic families of modular forms. Let $U = \overline{B}(k, p^{-r})$ ($r \geq 1$) denote the closed disk about k of radius p^{-r} in the weight space \mathcal{W} . We consider U as an affinoid space. The ring $\mathcal{O}(U)$ of analytic functions on U is isomorphic to the Tate algebra $A = E\left\{\left\{\frac{w}{p^r}\right\}\right\}$. Define

$$\kappa(w) = k + \frac{\log_p(1+w)}{\log_p(1+p)} \in A.$$

Then $w = (1+p)^{\kappa(w)-k} - 1$. For each $f \in A$ we define $\mathcal{A}^{\text{wt}}(f) \in E[[\kappa - k]]$ by setting $\mathcal{A}^{\text{wt}}(f) := f((1+p)^{\kappa-k} - 1)$. We also set $\varpi_\kappa := \frac{\log_p(1+w)}{\log_p(1+p)} \in A$ so that $\mathcal{A}^{\text{wt}}(\varpi_\kappa) = \kappa - k$.

Recall that \mathcal{H} denotes the ring of formal power series $f(X) \in E[[X]]$ which converge on the open unit disk. Fix a generator $\gamma \in \Gamma_0$ and set

$$s = \frac{\log_p(1+X)}{\log_p \chi(\gamma)}.$$

Then $X = \chi(\gamma)^s - 1$ and for each $f(X) \in \mathcal{H}$ we set $\mathcal{A}^{\text{cyc}}(f) = f(\chi(\gamma)^{-s} - 1)$.

Remark 4.5. The pairing given as the compositum of the arrows

$$\tilde{H}_f^1(V_f) \times \tilde{H}_f^1(V_f) \xrightarrow{j^{-1}} \tilde{H}_f^1(V_f) \times \tilde{H}_f^1(V_f^*(1)) \xrightarrow{h_{V,D}} J_E/J_E^2 \xrightarrow{\mathcal{A}^{\text{cyc}}} E$$

is the pairing $-\mathfrak{h}_p$.

The transformations \mathcal{A}^{wt} and \mathcal{A}^{cyc} induce a map $\mathcal{A} : \mathcal{H}_A \rightarrow E[[\kappa, s]]$ which we call the two-variable Amice transform. Let $\chi : \Gamma_0 \rightarrow \mathcal{H}_A^\times$ be the character given by

$$\chi(\tau) = \chi(\tau)^{\varpi_\kappa}, \quad \tau \in \Gamma.$$

For each $\kappa \in k + p^{r-1}\mathbb{Z}_p$, we shall denote by ψ_κ the morphism

$$\begin{aligned} \psi_\kappa : A &\longrightarrow E \\ w &\longmapsto (1+p)^{\kappa-k} - 1. \end{aligned}$$

Set $I = \{\kappa \in \mathbb{Z}_{\geq 2} \mid \kappa \equiv k \pmod{(p-1)p^{r-1}}\}$ and let

$$\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in A[[q]]$$

be a p -adic family of cuspidal eigenforms passing through f . This means that for every point $\kappa \in I$ the series $\mathbf{f}_\kappa = \sum_{n=1}^{\infty} \psi_\kappa(\mathbf{a}_n) q^n$ is the q -expansion of a weight κ eigenform on $\Gamma_0(pN)$, and $\mathbf{f}_k = f$. On shrinking $\overline{B}(k, p^{-r})$ if necessary and using [16, Corollary B5.7.1], we may assume that \mathbf{f} is a family of constant slope $k/2 - 1$.

Let $W_{\mathbf{f}}$ denote the big Galois representation associated to the family \mathbf{f} with coefficients in $A = \mathcal{O}(U)$. Set $V_{\mathbf{f}} = W_{\mathbf{f}}(k/2)$. We have a skew-symmetric pairing

$$(4.6) \quad V_{\mathbf{f}} \times V_{\mathbf{f}} \rightarrow A(\chi\chi^{-1}).$$

In particular, the representation $V_{\mathbf{f}}(\chi^{1/2})$ is self-dual.

Remark 4.6. Let $\Theta := \frac{\gamma - \chi^{1/2}(\gamma^{-1})}{\log \chi(\gamma)} \in \mathcal{H}_A$ and let I_Θ denote the ideal principal ideal of \mathcal{H}_A generated by Θ . We then have the following natural isomorphism of Galois modules:

$$\overline{V}_{\mathbf{f}}/I_\Theta \cdot \overline{V}_{\mathbf{f}} \cong V_{\mathbf{f}}(\chi^{1/2}).$$

We remark that we have γ^{-1} in the definition of Θ (as opposed to γ itself) due to our definition of the Galois action on \mathcal{H}_A^ι .

Set $V_\kappa = V_{\mathbf{f}} \otimes_{A, \psi_\kappa} E$. When κ is a positive integer we have $V_\kappa = W_{\mathbf{f}_\kappa}(k/2)$, where $W_{\mathbf{f}_\kappa}$ is the p -adic representation associated to \mathbf{f}_κ by Deligne. According to Kisin, there exists an analytic function $\alpha(\kappa) \in A$ with values in E such that

- $\alpha(k) = \psi_k(\mathbf{a}_p) p^{-k/2}$;
- $\mathbb{D}_{\text{cris}}(V_\kappa)^{\varphi=\alpha(\kappa)}$ is a one-dimensional E -vector space for each $\kappa \in I$.

The second condition implies that V_κ is trianguline and therefore semistable at all $\kappa \in I$. Let $\alpha_\kappa = \psi_\kappa(\alpha)$ and β_κ denote the eigenvalues of φ acting on $\mathbb{D}_{\text{st}}(V_\kappa)$. Since the Hodge weights of V_κ are $(-k/2, \kappa - k/2 - 1)$, from the weak admissibility of $\mathbb{D}_{\text{st}}(V_\kappa)$ it follows that

$$v_p(\alpha_\kappa) + v_p(\beta_\kappa) = \kappa - k - 1, \quad \kappa \in I.$$

Since $v_p(\alpha_\kappa) = v_p(\alpha(0)) = -1$, we have $v_p(\beta_\kappa) = \kappa - k$. This implies that V_κ is crystalline for all $\kappa \in I \setminus \{k\}$. By [25, Theorem 0.3.4], this data defines a triangulation of the (φ, Γ) -module $\mathbb{D}_{\text{rig}, A}^\dagger(V_{\mathbf{f}})$. More precisely, $\mathbb{D}_{\text{rig}, A}^\dagger(V_{\mathbf{f}})$ sits in an exact sequence

$$0 \rightarrow \mathbb{D}_{\mathbf{f}} \rightarrow \mathbb{D}_{\text{rig}, A}^\dagger(V_{\mathbf{f}}) \rightarrow \widetilde{\mathbb{D}}_{\mathbf{f}} \rightarrow 0,$$

where $\mathbb{D}_{\mathbf{f}} = \mathcal{R}_A \cdot e_\delta$ and $\widetilde{\mathbb{D}}_{\mathbf{f}} = \mathcal{R}_A \cdot e_{\widetilde{\delta}}$, are (φ, Γ) -modules of rank 1 defined by characters $\delta : \mathbb{Q}_p^* \rightarrow A^*$ and $\widetilde{\delta} : \mathbb{Q}_p^* \rightarrow A^*$ such that

$$\begin{aligned} \delta(u) &= u^{k/2}, & \delta(p) &= p^{k/2} \alpha(w) \\ \widetilde{\delta}(u) &= u^{k/2+1-\kappa(w)}, & \widetilde{\delta}(p) &= p^{-k/2} \alpha^{-1}(w). \end{aligned}$$

Note that $\psi_k(\delta) = \delta$, $\psi_k(\widetilde{\delta}) = \widetilde{\delta}$ and $\mathbb{D}_f = \mathbb{D}_{\mathbf{f}} \otimes_{A, \psi_k} E$.

Theorem 4.7 (Stevens, Coleman–Iovita). *Assume that*

$$a_p = \mathbf{a}_p(k) = p^{k/2-1}.$$

Then for the Fontaine–Mazur \mathcal{L} -invariant we have

$$\mathcal{L}_{\text{FM}}(f) = -2p \cdot \alpha'(k),$$

where $\alpha(\kappa) = p^{-k/2} \mathbf{a}_p(\kappa)$.

Proof. Since $\mathcal{L}_{\text{C}}(f) = \mathcal{L}_{\text{FM}}(f)$ by [17], the theorem follows from [40], where such a formula was proved for Coleman's \mathcal{L} -invariant $\mathcal{L}_{\text{C}}(f)$. The first direct proof of this formula for Fontaine–Mazur's \mathcal{L} -invariant was discovered by Colmez [18]. Another proof based on the theory of (φ, Γ) -modules may be found in [2]. \square

Let $\mathfrak{p} = \ker(\psi_k)$ be the prime of A corresponding to the eigenform f . We call *central critical weight height pairing* $h_{\mathbf{f}}^{\text{c-wt}}$ the pairing h^{wt} given via the general theory in Section 3.3 for the family $V_{\mathbf{f}}(\chi^{1/2})$ that is equipped with the triangulation $\mathbb{D}_{\mathbf{f}}(\chi^{1/2})$:

$$h_{\mathbf{f}}^{\text{c-wt}} = h_{V_{\psi_k(\mathbf{f})}, \mathbb{D}_{\psi_k(\mathbf{f})}}^{\text{wt}} : \widetilde{H}_{\mathbf{f}}^1(V_f) \times \widetilde{H}_{\mathbf{f}}^1(V_f^*(1)) \rightarrow \mathfrak{p}/\mathfrak{p}^2.$$

Denote by

$$(4.7) \quad \mathbb{H}_{\mathbf{f}}^{\text{c-wt}} : \widetilde{H}_{\mathbf{f}}^1(V_f) \times \widetilde{H}_{\mathbf{f}}^1(V_f) \rightarrow \mathfrak{p}/\mathfrak{p}^2$$

the composition of the pairing $h_{\mathbf{f}}^{\text{c-wt}}$ with the isomorphism $H_{\mathbf{f}}^1(V_f) \simeq H_{\mathbf{f}}^1(V_f^*(1))$ that is induced by the skew-symmetric pairing $V_f \times V_f \rightarrow E(1)$. Since $(V_{\mathbf{f}}(\chi^{1/2}), \mathbb{D}_{\mathbf{f}}(\chi^{1/2}))$ is self-dual and the pairing $h_{\mathbf{f}}^{\text{c-wt}}$ is symmetric, we conclude that the pairing $\mathbb{H}_{\mathbf{f}}^{\text{c-wt}}$ is skew-symmetric:

$$(4.8) \quad \mathbb{H}_{\mathbf{f}}^{\text{c-wt}}([x_{\mathbf{f}}], [y_{\mathbf{f}}]) = -\mathbb{H}_{\mathbf{f}}^{\text{c-wt}}([y_{\mathbf{f}}], [x_{\mathbf{f}}]).$$

Definition 4.8. We denote by

$$\mathfrak{h}_{\mathbf{f}}^{\text{c-wt}} : \widetilde{H}_{\mathbf{f}}^1(V_f) \times \widetilde{H}_{\mathbf{f}}^1(V_f) \rightarrow E$$

the pairing defined via

$$\mathcal{A}(\mathbb{H}_{\mathbf{f}}^{\text{c-wt}}([x_{\mathbf{f}}], [y_{\mathbf{f}}])) = \mathfrak{h}_{\mathbf{f}}^{\text{c-wt}}([x_{\mathbf{f}}], [y_{\mathbf{f}}])(\kappa - k).$$

4.3. Two-variable p -adic heights. In this section, we construct infinitesimal deformations of Nekovář's p -adic heights along the weight direction. Our construction is a direct generalization of Venerucci's two-variable height pairing. We keep notation and conventions of Section 4.2.

Let $\overline{V}_{\mathbf{f}} = V_{\mathbf{f}} \widehat{\otimes}_A \mathcal{H}_A^{\iota}$ and $\overline{\mathbb{D}}_{\mathbf{f}} = \mathbb{D}_{\mathbf{f}} \otimes_{\mathcal{R}_A} \mathbb{D}_{\text{rig}, A}^{\dagger}(\mathcal{H}_A^{\iota})$ (see Section 2.4). Let \mathfrak{P} denote the kernel of the augmentation map

$$\begin{aligned} \mathcal{H}_A &\longrightarrow E \\ f(X) &\mapsto \psi_k(f(0)) \end{aligned}$$

Consider the tautological exact sequence

$$0 \rightarrow \mathfrak{P}/\mathfrak{P}^2 \rightarrow \mathcal{H}_A/\mathfrak{P}^2 \rightarrow E \rightarrow 0.$$

Tensoring this exact sequence with $\mathbf{R}\Gamma(\overline{V}_{\mathbf{f}}, \overline{\mathbb{D}}_{\mathbf{f}})$ we get a distinguished triangle

$$\mathbf{R}\Gamma(\overline{V}_{\mathbf{f}}, \overline{\mathbb{D}}_{\mathbf{f}}) \otimes_{\mathcal{H}_A}^{\mathbb{L}} \mathfrak{P}/\mathfrak{P}^2 \rightarrow \mathbf{R}\Gamma(\overline{V}_{\mathbf{f}}, \overline{\mathbb{D}}_{\mathbf{f}}) \otimes_{\mathcal{H}_A}^{\mathbb{L}} \mathcal{H}_A/\mathfrak{P}^2 \rightarrow \mathbf{R}\Gamma(\overline{V}_{\mathbf{f}}, \overline{\mathbb{D}}_{\mathbf{f}}) \otimes_{\mathcal{H}_A}^{\mathbb{L}} E.$$

By the base change theorem for Selmer complexes (c.f., [35], Section 1), we have a canonical isomorphism

$$\mathbf{R}\Gamma(\overline{V}_{\mathbf{f}}, \overline{\mathbb{D}}_{\mathbf{f}}) \otimes_{\mathcal{H}_A}^{\mathbb{L}} E \xrightarrow{\sim} \mathbf{R}\Gamma(V_f, \mathbb{D}_f).$$

Using the natural identification

$$\mathbf{R}\Gamma(\overline{V}_{\mathbf{f}}, \overline{\mathbb{D}}_{\mathbf{f}}) \otimes_{\mathcal{H}_A}^{\mathbb{L}} \mathfrak{P}/\mathfrak{P}^2 \simeq \mathbf{R}\Gamma(V_f, \mathbb{D}) \otimes_E \mathfrak{P}/\mathfrak{P}^2,$$

this distinguished triangle translates to

$$\mathbf{R}\Gamma(V_f, \mathbb{D}_f) \otimes_E \mathfrak{P}/\mathfrak{P}^2 \rightarrow \mathbf{R}\Gamma(\overline{V}_{\mathbf{f}}, \overline{\mathbb{D}}_{\mathbf{f}}) \otimes_{\mathcal{H}_A}^{\mathbb{L}} \mathcal{H}_A/\mathfrak{P}^2 \rightarrow \mathbf{R}\Gamma(V_f, \mathbb{D}_f) \xrightarrow{\beta} \mathbf{R}\Gamma(V_f, \mathbb{D}_f)[1] \otimes_E \mathfrak{P}/\mathfrak{P}^2.$$

On the level of cohomology, we have a map

$$\tilde{H}_{\mathbf{f}}^1(V_f) \rightarrow \tilde{H}_{\mathbf{f}}^2(V_f) \otimes_E \mathfrak{P}/\mathfrak{P}^2$$

which we denote again by β .

Definition 4.9. The *two-variable height pairing* $\mathbb{H}_{\mathbf{f}}$ is defined as the compositum of the following arrows:

$$\tilde{H}_{\mathbf{f}}^1(V_f) \otimes_E \tilde{H}_{\mathbf{f}}^1(V_f) \xrightarrow{\beta \otimes \text{id}} \left(\tilde{H}_{\mathbf{f}}^2(V_f) \otimes_E \mathfrak{P}/\mathfrak{P}^2 \right) \otimes_E \tilde{H}_{\mathbf{f}}^1(V_f^*(1)) \xrightarrow{\cup_{V_f, \mathbb{D}_f}} \mathfrak{P}/\mathfrak{P}^2$$

where \cup_{V_f, \mathbb{D}_f} is the cup-product (2.9).

Let \mathfrak{I} denote the ideal of the ring $E[[\kappa - k, s]]$ generated by $\kappa - k$ and s . The map \mathcal{A} identifies $\mathfrak{P}/\mathfrak{P}^2$ with $\mathfrak{I}/\mathfrak{I}^2$.

The main formal properties of the pairing $\mathbb{H}_{\mathbf{f}}$ are listed in the following theorem.

Theorem 4.10. *For any $[x_{\mathbf{f}}], [y_{\mathbf{f}}] \in \tilde{H}_{\mathbf{f}}^1(V_f)$, we have the following identities in $\mathfrak{I}/\mathfrak{I}^2$.*

- i) $\frac{\partial}{\partial s} \mathcal{A}(\mathbb{H}_{\mathbf{f}}([x_{\mathbf{f}}], [y_{\mathbf{f}}])) \big|_{s=0, \kappa=k} = -\mathfrak{h}_p([x_{\mathbf{f}}], [y_{\mathbf{f}}])$ and $\frac{\partial}{\partial \kappa} \mathcal{A}(\mathbb{H}_{\mathbf{f}}([x_{\mathbf{f}}], [y_{\mathbf{f}}]))(\kappa, (\kappa - k)/2) \big|_{\kappa=k} = \mathfrak{h}_{\mathbf{f}}^{\text{c-wt}}([x_{\mathbf{f}}], [y_{\mathbf{f}}])$.
- ii) $\mathcal{A}(\mathbb{H}_{\mathbf{f}}([y_{\mathbf{f}}], [x_{\mathbf{f}}]))(\kappa, s) = -\mathcal{A}(\mathbb{H}_{\mathbf{f}}([x_{\mathbf{f}}], [y_{\mathbf{f}}]))(\kappa, \kappa - k - s)$.

iii) (*Rubin-style formulae, Part 1*) Assume that $a_p = p^{k/2-1}$ and let $[x_{\mathbf{f}}] \in \tilde{H}_{\mathbf{f}}^1(V_f)$ denote the lift of a class $[x] \in H_{\mathbf{f}}^1(\mathbb{Q}, V_f)$ with respect to the canonical splitting spl . Suppose

that the class $[x_f]$ is represented by $(x, (x_\ell^+), (\lambda_\ell))$. We then have the following identity in $\mathfrak{J}/\mathfrak{J}^2$:

$$(4.9) \quad \mathcal{A}(\mathbb{H}_f([x_f], \partial_0(d_{\tilde{\delta}}))) = \langle \Psi_2, [x_p^+] \rangle \cdot s.$$

Furthermore,

$$(4.10) \quad \mathcal{A}(\mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), \partial_0(d_{\tilde{\delta}}))) = \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \cdot \left(s - \frac{\kappa - k}{2}\right).$$

Proof. i) These two identities follow directly from definitions, see in particular [41, Section 0.22 in Appendix C] for a general formalism in the p -ordinary setting. We remark that the sign in the first identity is due to our normalization of the cyclotomic Amice transform; see Remark 4.5.

ii) Since $\mathcal{A}(\mathbb{H}_f([y_f], [x_f]))(\kappa, s)$ is a linear form, it is sufficient to prove that (ii) holds for $\kappa = k$ and $s = (\kappa - k)/2$. For $\kappa = k$, we have, by (i) and the symmetry of \mathfrak{h}_p

$$\mathcal{A}(\mathbb{H}_f([y_f], [x_f]))(k, s) = -\mathfrak{h}_p([y_f], [x_f]) = -\mathfrak{h}_p([x_f], [y_f]) = -\mathcal{A}(\mathbb{H}_f([x_f], [y_f]))(k, -s).$$

Using (i) we have

$$\mathcal{A}(\mathbb{H}_f([x_f], [y_f])) \Big|_{s=\frac{\kappa-k}{2}} = \mathfrak{h}_f^{\text{c-wt}}([x_f], [y_f]) \cdot (\kappa - k).$$

Since the bilinear pairing $\mathfrak{h}_f^{\text{c-wt}}$ is skew-symmetric,

$$\mathcal{A}(\mathbb{H}_f([x_f], [y_f]))(\kappa, (\kappa - k)/2) = -\mathcal{A}(\mathbb{H}_f([y_f], [x_f]))(\kappa, (\kappa - k)/2)$$

as desired.

iii) Set $d_{\tilde{\delta}} := e_{\tilde{\delta}} \otimes t^{k/2-1}$ so that $d_{\tilde{\delta}}$ spans the E -vector space $H^0(\tilde{\mathbb{D}}_f)$. Choose any lift $z \in \mathbb{D}_{\text{rig}}^\dagger(V_f)$ of $d_{\tilde{\delta}}$ under the canonical projection of $\mathbb{D}_{\text{rig}}^\dagger(V_f)$ onto $\tilde{\mathbb{D}}_f$. Then the class $\partial_0(d_{\tilde{\delta}}) \in H^1(\tilde{\mathbb{D}}_f)$ may be represented by the cocycle

$$(0, (a_\ell^+), (\mu_\ell)) \in C^1(G_{\mathbb{Q}, S}, V_f) \oplus U_S^1(V_f, \mathbb{D}_f) \oplus K^0(V_f),$$

such that $a_\ell^+ = \mu_\ell = 0$ for all $\ell \neq p$, and

$$\mu_p = z, \quad a_p^+ = ((\varphi - 1)z, (\gamma - 1)z).$$

Let $\mathbf{z} \in C_{\varphi, \gamma}^0(V_f) = \mathbb{D}_{\text{rig}, A}^\dagger(V_f)$ be any lift of z under the projection induced by the augmentation map $\psi_k : A \rightarrow E$, whose kernel is the prime \mathfrak{p} associated to the form f . Then,

$$\mathbf{z} \otimes 1 \in \overline{\mathbb{D}_{\text{rig}, A}^\dagger(V_f)} := \mathbb{D}_{\text{rig}, A}^\dagger(V_f) \otimes_{\mathcal{R}_A} \mathbb{D}_{\text{rig}, A}^\dagger(\mathcal{H}_A^\iota)$$

is a lift of $d_{\tilde{\delta}}$ under the projection

$$\overline{\mathbb{D}_{\text{rig}, A}^\dagger(V_f)} \longrightarrow \tilde{\mathbb{D}}_f \otimes_{\mathcal{R}_A} \mathbb{D}_{\text{rig}, A}^\dagger(\mathcal{H}_A^\iota).$$

Setting $\mathbf{a}_\ell^+ = \boldsymbol{\mu}_\ell = 0$ for all $\ell \neq p$ and

$$\boldsymbol{\mu}_p = (z \otimes 1), \quad \mathbf{a}_p^+ = ((\varphi - 1)\mathbf{z}, (\gamma - 1)(\mathbf{z} \otimes 1))$$

we see that $(0, (\mathbf{a}_\ell^+), (\boldsymbol{\mu}_\ell))$ is a cochain in $S^1(\overline{V}_f, \overline{\mathbb{D}}_f)$ which lifts $\partial_0(d_{\tilde{\delta}})$. We therefore infer that $\beta(\partial_0(d_{\tilde{\delta}}))$ is the class of the differential

$$d(0, (\mathbf{a}_\ell^+), (\boldsymbol{\mu}_\ell)) = (0, \star, (\mathbf{v}_\ell)),$$

where $\mathbf{v}_\ell = 0$ for all $\ell \neq p$ and $\mathbf{v}_p = -((\varphi - 1)(\mathbf{z} \otimes 1), (\gamma - 1)(\mathbf{z} \otimes 1))$. Let $[x_f] = [x, (x_\ell^+), (\lambda_\ell)] \in H_f^1(\mathbb{Q}, V_f)$. Then by the definition of the cup-product given by (2.8),

$$\mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), [x_f]) = \mathbf{v}_p \cup x_p^+ = \tilde{\mathbf{v}}_p \cup x_p^+,$$

where $\tilde{\mathbf{v}}_p$ denotes the image of \mathbf{v}_p in $C^1(\tilde{\mathbb{D}}_f \hat{\otimes}_{\mathcal{R}_A} \mathbb{D}_{\text{rig}, A}^\dagger(\mathcal{H}_A^\iota))$ under the natural projection and it is explicitly given by the formula

$$\tilde{\mathbf{v}}_p = -((\varphi - 1)(e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1), (\gamma - 1)(e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1)).$$

Since $\varphi(e_{\tilde{\delta}}) = p^{-k/2}\alpha^{-1}(\kappa)e_{\tilde{\delta}}$ and

$$\frac{1}{\alpha(\kappa)} \equiv \frac{1}{\alpha(k)} - \frac{\alpha'(k)}{\alpha^2(k)} \cdot (\kappa - k) \pmod{\mathfrak{I}^2},$$

we have

$$\mathcal{A}((\varphi - 1)(e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1)) \equiv -p\alpha'(k) \cdot (\kappa - k) \cdot d_{\tilde{\delta}} \pmod{\mathfrak{I}^2}.$$

Taking into account the statement of Theorem 4.7, we deduce that

$$\mathcal{A}((\varphi - 1)(e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1)) = \frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot (\kappa - k) \cdot d_{\tilde{\delta}} \pmod{\mathfrak{I}^2}.$$

On the other hand, since $\gamma(e_{\tilde{\delta}}) = \chi(\gamma)^{k/2+1-\kappa}e_{\tilde{\delta}}$, we have

$$\begin{aligned} \mathcal{A}((\gamma - 1)(e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1)) &= \mathcal{A}(\chi(\gamma)^{k-\kappa} \cdot e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes \gamma^{-1} - e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1) \\ &= \mathcal{A}(\chi(\gamma)^{k-\kappa} \cdot e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes (\gamma^{-1} - 1)) \\ &\quad + \mathcal{A}((\chi(\gamma)^{k-\kappa} - 1) \cdot e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1) \\ &\equiv (\log \chi(\gamma) \cdot (k - \kappa) + 1) \cdot \log \chi(\gamma) \cdot s \cdot e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1 \\ &\quad + \log \chi(\gamma)(k - \kappa) \cdot e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1 \\ &\equiv s \log \chi(\gamma) \cdot (e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1) \\ &\quad + \log \chi(\gamma) \cdot (k - \kappa) \cdot (e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1) \\ &\equiv (s + k - \kappa) \log \chi(\gamma) \cdot (e_{\tilde{\delta}} \otimes t^{k/2-1} \otimes 1) \\ &\equiv (s + k - \kappa) \log \chi(\gamma) \cdot d_{\tilde{\delta}} \pmod{\mathfrak{I}^2}. \end{aligned}$$

To summarize, we just verified that

$$\mathcal{A}(\tilde{\mathbf{v}}_p) = \left(\frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot (\kappa - k), \log \chi(\gamma) \cdot (s + k - \kappa) \right) d_{\tilde{\delta}}$$

and for the class of $\tilde{\mathbf{v}}_p$ in $H^1(\tilde{\mathbb{D}}_f) \otimes_E \mathfrak{P}/\mathfrak{P}^2$ we have

$$\mathcal{A}([\tilde{\mathbf{v}}_p]) = -\frac{\mathcal{L}_{\text{FM}}(f)}{2}(\kappa - k)\Psi_1 + (s + k - \kappa)\Psi_2 \pmod{\mathfrak{I}^2}.$$

We are now ready to prove our formulas. First assume that $[x_f] = [x, (x_\ell^+), (\lambda_\ell)]$ is the canonical lift of some $[x] \in H_f^1(\mathbb{Q}, V_f)$. Then $[x_p^+] \in H_f^1(\mathbb{D}_f)$ and we have

$$\mathcal{A}(\mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), [x_f])) = \langle \Psi_2, [x_p^+] \rangle \cdot (s + k - \kappa).$$

Therefore,

$$\mathcal{A}(\mathbb{H}_f([x_f], \partial_0(d_{\tilde{\delta}}))) = \langle \Psi_2, [x_p^+] \rangle \cdot s.$$

We now prove the second formula. Write $\partial_0(d_{\tilde{\delta}}) = a\Psi_1^* + b\Psi_2^*$. Then $\mathcal{L}_{\text{FM}}(f) = b/a$ and

$$\begin{aligned} \mathcal{A}(\mathbb{H}_{\mathbf{f}}(\partial_0(d_{\tilde{\delta}}), \partial_0(d_{\tilde{\delta}}))) &= a \cdot \frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot (\kappa - k) + b \cdot (s + k - \kappa) = \\ &= b \cdot \left(s - \frac{1}{2}(\kappa - k) \right) = \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \cdot \left(s - \frac{1}{2}(\kappa - k) \right). \end{aligned}$$

□

We remark that the identities in Theorem 4.10(i) are equivalent to saying that

$$(4.11) \quad \mathcal{A}(\mathbb{H}_{\mathbf{f}}([x_{\mathbf{f}}], [y_{\mathbf{f}}]))(\kappa, s) = -\mathfrak{h}_p([x_{\mathbf{f}}], [y_{\mathbf{f}}]) \cdot \left(s - \frac{\kappa - k}{2} \right) + \mathfrak{h}_{\mathbf{f}}^{\text{c-wt}}([x_{\mathbf{f}}], [y_{\mathbf{f}}]) \cdot (\kappa - k)$$

4.4. Rubin-style Formulae, Part II. Throughout this section, we shall be in one of the following two settings (that we will simultaneously treat):

(A) We have $\mathfrak{B} = \mathcal{H}_E$ and $\mathfrak{J} = J_E$, where $J_E = X \cdot \mathcal{H}_E$ is the kernel of the augmentation map and $X = \gamma - 1$. We set $V = \overline{V}_f = V_f \otimes \mathcal{H}_E^{\iota}$ and $\mathbb{D} = \overline{\mathbb{D}}_f = \mathbb{D}_f \otimes_{\mathcal{R}} \mathbb{D}_{\text{rig}}^{\dagger}(\mathcal{H}_E^{\iota})$. The (φ, Γ) -module \mathbb{D} is a submodule of the (φ, Γ) -module $\mathbb{D}_{\text{rig}, \mathfrak{B}}^{\dagger}(V)$ associated V and we set $\widetilde{\mathbb{D}} = \mathbb{D}_{\text{rig}, \mathfrak{B}}^{\dagger}(V)/\mathbb{D}$ (and it also equals $\widetilde{\mathbb{D}}_f \otimes \mathbb{D}_{\text{rig}}^{\dagger}(\mathcal{H}_E^{\iota})$). Observe that we have $V/\mathfrak{J}V \cong V_f$ (and a similar isomorphism for all relevant (φ, Γ) -modules). Furthermore, we have the symmetric p -adic height pairing

$$\mathfrak{H} : \widetilde{H}_{\mathbf{f}}^1(V_f) \times \widetilde{H}_{\mathbf{f}}^1(V_f) \longrightarrow \mathfrak{J}/\mathfrak{J}^2$$

induced from the pairing (4.4) via the isomorphism $V_f \cong V_f^*(1)$.

(B) We have $\mathfrak{B} = A$ where A is an affinoid domain as in Section 4.2. In this set up, we will consider the Galois representation $V := V_{\mathbf{f}}(\chi^{1/2})$, which is the central critical twist of the big Galois representation associated to the family \mathbf{f} . This representation is equipped with a triangulation, which is given by the (φ, Γ) -submodule $\mathbb{D} := \mathbb{D}_{\mathbf{f}}(\chi^{1/2})$. As above, we will also let $\widetilde{\mathbb{D}} := \mathbb{D}_{\text{rig}, A}^{\dagger}(V)/\mathbb{D}$. The prime \mathfrak{p} is that corresponds to the form f we have fixed at the start and we set $\mathfrak{J} = \mathfrak{p}$. Observe that again $V/\mathfrak{J}V \cong V_f$ (and a similar isomorphism for all relevant (φ, Γ) -modules). Furthermore, we have an *anti-symmetric* p -adic height pairing

$$\mathfrak{H} := \mathbb{H}_{\mathbf{f}}^{\text{c-wt}} : \widetilde{H}_{\mathbf{f}}^1(V_f) \times \widetilde{H}_{\mathbf{f}}^1(V_f) \longrightarrow \mathfrak{p}/\mathfrak{p}^2$$

which is given by (4.7).

Definition 4.11. In the situation of (A) or (B) set $V_{\varepsilon} = V \otimes \mathfrak{B}/\mathfrak{J}^2$, $\mathbb{D}_{\varepsilon} := \mathbb{D} \otimes \mathfrak{B}/\mathfrak{J}^2$ and $\widetilde{\mathbb{D}}_{\varepsilon} = \widetilde{\mathbb{D}} \otimes \mathfrak{B}/\mathfrak{J}^2$.

Note that these objects also make their appearance in Sections 2.6.1 and 3.2.1 of [6], where V_{ε} is denoted by \tilde{V} , etc.

In the situation of (A) or (B), note that we have the following exact sequence

$$(4.12) \quad 0 \longrightarrow V_f \otimes \mathfrak{J}/\mathfrak{J}^2 \longrightarrow V_{\varepsilon} \longrightarrow V_f \longrightarrow 0$$

as well as the following tautological exact triangle in the derived category:

$$(4.13) \quad \mathbf{R}\Gamma(G_p, \mathbb{D}_{\varepsilon}) \longrightarrow \mathbf{R}\Gamma(G_p, \mathbb{D}_{\text{rig}}^{\dagger}(V_{\varepsilon})) \xrightarrow{s} \mathbf{R}\Gamma(G_p, \widetilde{\mathbb{D}}_{\varepsilon}) \longrightarrow \mathbf{R}\Gamma(G_p, \mathbb{D}_{\varepsilon})[1]$$

where $? = f, \varepsilon$. The quasi-isomorphism α of Proposition 2.2 together with the sequence (2.7) and (4.13) induces a functorial quasi-isomorphism

$$(4.14) \quad \mathbf{R}\Gamma(G_p, \tilde{\mathbb{D}}_?) \xrightarrow{qis} \tilde{U}_p(V, \mathbb{D}_?).$$

which is compatible with all duality statements in the obvious sense. By the local-global compatibility of the Langlands correspondence (see [13]) it follows that

$$(4.15) \quad H^0(G_\ell, V_f) = H_f^1(G_\ell, V_f) = 0, \quad \ell \neq p.$$

In particular,

$$\mathbf{R}\Gamma(G_\ell, V_?) \xrightarrow{qis} \tilde{U}_\ell(V_?, \mathbb{D}_?), \quad \ell \neq p.$$

To simplify notation, we set $\tilde{U}_S(V_?) = \bigoplus_{\ell \in S} \tilde{U}_\ell(V_?, \mathbb{D}_?)$.

The exact sequence (4.12) induces a commutative diagram of complexes

$$(4.16) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{C}_f^\bullet(V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 & \longrightarrow & C^\bullet(G_{\mathbb{Q},S}, V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 & \xrightarrow{\widetilde{\text{res}}_S} & \tilde{U}_S(V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{C}_f^\bullet(V_\varepsilon) & \longrightarrow & C^\bullet(G_{\mathbb{Q},S}, V_\varepsilon) & \xrightarrow{\widetilde{\text{res}}_S} & \tilde{U}_S(V_\varepsilon) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{C}_f^\bullet(V_f) & \longrightarrow & C^\bullet(G_{\mathbb{Q},S}, V_f) & \xrightarrow{\widetilde{\text{res}}_S} & \tilde{U}_S(V_f) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where we recall that $\tilde{C}_f^\bullet(V_?)$ is the shorthand for the Selmer complex $S^\bullet(V_?, \mathbb{D}_?)$ (for $? = f, \varepsilon$).

In the level of cohomology, the diagram (4.16) induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & & & & & H^0(\tilde{U}_S(V_f)) \\ & & & & & & \downarrow \beta^0 \\ & & & & & & H^1(\tilde{U}_S(V_f)) \otimes \mathfrak{I}/\mathfrak{I}^2 \\ & & & & & & \downarrow i \\ & & & & H^1(G_{\mathbb{Q},S}, V_\varepsilon) & \xrightarrow{\widetilde{\text{res}}_S} & H^1(\tilde{U}_S(V_\varepsilon)) \\ & & & & \downarrow \text{pr} & & \downarrow \text{pr} \\ H^0(\tilde{U}_S(V_f)) & \longrightarrow & \tilde{H}_f^1(V_f) & \longrightarrow & H^1(G_{\mathbb{Q},S}, V_f) & \xrightarrow{\widetilde{\text{res}}_S} & H^1(\tilde{U}_S(V_f)) \\ & & \downarrow \beta^1 & & \downarrow & & \downarrow \\ & & & & H^2(G_{\mathbb{Q},S}, V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 & \longrightarrow & H^2(\tilde{U}_S(V_f)) \otimes \mathfrak{I}/\mathfrak{I}^2 \\ & & & & \downarrow & & \\ H^1(\tilde{U}_S(V_f)) \otimes \mathfrak{I}/\mathfrak{I}^2 & \xrightarrow{\partial_1} & \tilde{H}_f^2(V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 & \longrightarrow & H^2(G_{\mathbb{Q},S}, V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 & \longrightarrow & H^2(\tilde{U}_S(V_f)) \otimes \mathfrak{I}/\mathfrak{I}^2 \end{array}$$

where pr is the map induced from the exact sequence (4.12).

Lemma 4.12. *Suppose that we are given a class $[x_f] = [(x, (x_\ell^+)_{\ell \in S}, (\lambda_\ell)_{\ell \in S})] \in \tilde{H}_f^1(V_f)$ such that $\text{pr}_0([\mathbb{X}]) = [x] \in H_f^1(\mathbb{Q}, V_f)$ for some $[\mathbb{X}] \in H^1(G_{\mathbb{Q}, S}, V_\varepsilon)$. Then there exists a class*

$$[\mathfrak{D}\mathbb{X}] = ([\mathfrak{D}\mathbb{X}]_\ell)_{\ell \in S} \in H^1(\tilde{U}_S(V_f)) \otimes \mathfrak{J}/\mathfrak{J}^2$$

such that

$$(i) \ i([\mathfrak{D}\mathbb{X}]) = \widetilde{\text{res}}_S(\mathbb{X}) .$$

$$(ii) \ \beta^1([x_f]) = -\partial_1([\mathfrak{D}\mathbb{X}]) .$$

Proof. Since we have

$$\text{pr} \circ \widetilde{\text{res}}_S([\mathbb{X}]) = \widetilde{\text{res}}_S([x]) = 0$$

it follows from Lemma 1.2.19 of [31] (or simply chasing the diagram above) that there exists a class $d \in H^1(\tilde{U}_S(V_f)) \otimes \mathfrak{J}/\mathfrak{J}^2$ that verifies

- $i(d) = \widetilde{\text{res}}_S([\mathbb{X}])$,
- $\beta^1([x_f]) + \partial_1(d) - \partial_1 \circ \beta^0(t) = 0$ for some $t \in H^0(\tilde{U}_S(V_f))$.

Set $[\mathfrak{D}\mathbb{X}] = d - \beta^0(t)$. □

The following statement is what we call the *analytic Rubin-style formula* for the height pairing \mathfrak{H} and the elliptic modular form f (in either of the situations (A) or (B)).

Theorem 4.13 (Analytic Rubin-style formula). *Let $[x_f] = [(x, (x_\ell^+), \lambda_\ell)]$ and $[y_f] = [(y, (y_\ell^+), \mu_\ell)] \in \tilde{H}_f^1(V_f)$ be two elements such that $[x] \in H_f^1(\mathbb{Q}, V_f)$. Suppose further that there is an element $[\mathbb{X}] \in H^1(G_{\mathbb{Q}, S}, V_\varepsilon)$ with the property that $\text{pr}_0([\mathbb{X}]) = [x]$. Then,*

$$\mathfrak{H}([x_f], [y_f]) = -\text{inv}_p(\mathfrak{D}\mathbb{X}_p \cup i_p^+(y_p^+))$$

where inv_p denotes the local invariant map at p and $\mathfrak{D}\mathbb{X}_p$ is any cocycle representing $[\mathfrak{D}\mathbb{X}]_p$.

Remark 4.14. Although the cocycle $\mathfrak{D}\mathbb{X}_p$ is defined only up to a coboundary $d(u, v) \in d\tilde{U}_S(V_f)^0 \otimes \mathfrak{J}/\mathfrak{J}^2$, it is not hard to see that the right hand side of this formula does not depend on its choice, i.e.,

$$\text{inv}_p(d(u, v) \cup y_p^+) = 0.$$

We refer the reader to [12, Remark A.3] where this is carried out in full detail in the p -ordinary case. In our setting, we simply need to appeal to the discussion [6, Section 1.2.1] in place of [31, Sections 1.3.1 and 6.2.2].

Definition 4.15. Set $\omega = X$ in the situation of (A) and $\omega = \varpi_\kappa$ in the situation of (B). We let $\mathfrak{d}[\mathbb{X}] \in H^1(\tilde{\mathbb{D}}_f)$ be the unique element such that $\mathfrak{d}[\mathbb{X}] \otimes \omega \pmod{\mathfrak{J}^2} \in H^1(\tilde{\mathbb{D}}_f) \otimes \mathfrak{J}/\mathfrak{J}^2$ corresponds to $[\mathfrak{D}\mathbb{X}]_p$ under the isomorphism

$$H^1(\tilde{U}_p(V_f, \mathbb{D}_f)) \otimes \mathfrak{J}/\mathfrak{J}^2 \simeq H^1(\tilde{\mathbb{D}}_f) \otimes \mathfrak{J}/\mathfrak{J}^2$$

induced from (4.14). The class $\mathfrak{d}[\mathbb{X}]$ will be called the *Bockstein-normalized derivative* of \mathbb{X} .

Corollary 4.16. *In the situation of Theorem 4.13 we have*

$$\mathfrak{H}([x_f], [y_f]) = - \langle \mathfrak{d} [\mathbb{X}], [y_p^+] \rangle \cdot \omega$$

where $\langle \cdot, \cdot \rangle : H^1(\widetilde{\mathbb{D}}_f) \times H^1(\mathbb{D}_f) \rightarrow E$ is the canonical pairing and $\omega = X$ when we are in the situation of **(A)** and $\omega = \varpi_\kappa$ in the situation of **(B)**.

Proof of Theorem 4.13. The proof of this theorem is purely formal and follows the proof of [31, Proposition 11.3.15] with obvious modifications. Let

$$Z^\bullet = \text{cone} \left(\tau_{\geq 2} C^\bullet(G_{\mathbb{Q}, S}, A(1)) \xrightarrow{\text{res}_S} \tau_{\geq 2} K_S^\bullet(A(1)) \right) [-1].$$

The global class field theory gives rise to the following diagram with exact rows:

$$\begin{array}{ccccccc} H^2(G_{\mathbb{Q}, S}, \mathbb{Q}_p(1)) \otimes \mathfrak{I}/\mathfrak{I}^2 & \xrightarrow{\text{res}_S} & \bigoplus_{\ell \in S} H^2(G_\ell, \mathbb{Q}_p(1)) \otimes \mathfrak{I}/\mathfrak{I}^2 & \rightarrow & H^3(Z^\bullet) \otimes \mathfrak{I}/\mathfrak{I}^2 & \rightarrow & 0 \\ \parallel & & \parallel & & \downarrow \text{inv}_S & & \\ H^2(G_{\mathbb{Q}, S}, \mathbb{Q}_p(1)) \otimes \mathfrak{I}/\mathfrak{I}^2 & \xrightarrow{\text{res}_S} & \bigoplus_{\ell \in S} H^2(G_\ell, \mathbb{Q}_p(1)) \otimes \mathfrak{I}/\mathfrak{I}^2 & \xrightarrow{\sum_{\ell \in S} \text{inv}_\ell} & \mathfrak{I}/\mathfrak{I}^2 & \rightarrow & 0 \end{array}$$

Suppose $[z_f] = [(z, z_S^+, \omega_S)] \in \widetilde{H}_f^2(V_f) \otimes \mathfrak{I}/\mathfrak{I}^2$ and $[y_f] = [(y, y_S^+, \mu_S)] \in \widetilde{H}_f^1(V_f)$, where we set $z_S^+ = (z_\ell^+)$, $\omega_S = (\omega_\ell)$, $y_S^+ = (y_\ell^+)$ and $\mu_S = (\mu_\ell)$ to simplify notation. The formula

$$z_f \widetilde{\cup} y_f := (z \cup y, \omega_S \cup \text{res}_S(y) + i_S^+(z_S^+) \cup \mu_S) \in Z^3 \otimes \mathfrak{I}/\mathfrak{I}^2$$

defines a cup product

$$\widetilde{\cup} : \left(\widetilde{C}_f^2(V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 \right) \otimes \widetilde{C}_f^1(V_f) \rightarrow Z^3 \otimes \mathfrak{I}/\mathfrak{I}^2$$

which is homotopic to the cup-product (2.8) by [31, Proposition 1.3.2]. Therefore the duality

$$\langle \cdot, \cdot \rangle_{\text{PT}} : \left(\widetilde{H}_f^2(V_f) \otimes \mathfrak{I}/\mathfrak{I}^2 \right) \otimes \widetilde{H}_f^1(V_f) \rightarrow H^3(Z^\bullet) \otimes \mathfrak{I}/\mathfrak{I}^2 \xrightarrow{\text{inv}_S} \mathfrak{I}/\mathfrak{I}^2$$

induced by (2.9) on the level of cohomology can be computed by

$$\langle [z_f], [y_f] \rangle_{\text{PT}} = \text{inv}_S(z_f \widetilde{\cup} y_f).$$

Since the cohomological dimension of G_S is 2, $x \cup y$ is a coboundary and there exists a cochain W such that $dW = z \cup y$. Therefore one may compute $\text{inv}_S([z_f \widetilde{\cup} y_f])$ to be

$$\text{inv}_S([z_f \widetilde{\cup} y_f]) = \sum_{\ell \in S} \text{inv}_\ell (\omega_\ell \cup \text{res}_\ell(y) + i_\ell^+(z_\ell^+) \cup \mu_\ell + \text{res}_\ell(W)).$$

Let now $[z_f] = \beta^1([x_f])$, where $[x_f] \in \widetilde{H}^1(V_f)$. Since $\text{pr}(\mathfrak{X}) = [x]$, from the exact sequence

$$H^1(G_{\mathbb{Q}, S}, V_\varepsilon) \rightarrow H^1(G_{\mathbb{Q}, S}, V_f) \xrightarrow{\beta^1} H^2(G_{\mathbb{Q}, S}, V_f)$$

it follows that z is a coboundary. Write $z = dA$ for some $A \in C^1(G_{\mathbb{Q}, S}, V_f)$ and take $W = A \cup y$. Then $i_\ell^+(z_\ell^+) = d(\omega_\ell + \text{res}_\ell(A))$ for each $\ell \in S$ and it is easy to check that

$$\beta^1([x_f]) = \partial \circ \widetilde{\text{res}}_S([(\omega_\ell + \text{res}_\ell(A))_\ell]).$$

Thus,

$$\begin{aligned}
\mathfrak{H}([x_f], [y_f]) &= \sum_{\ell \in S} \text{inv}_\ell (\omega_\ell \cup \text{res}_\ell(y) + i_\ell^+(z_\ell^+) \cup \mu_\ell + \text{res}_\ell(A) \cup \text{res}_\ell(y)) = \\
&= \sum_{\ell \in S} \text{inv}_\ell ((\omega_\ell + \text{res}_\ell(A)) \cup \text{res}_\ell(y) + d(\omega_\ell + \text{res}_\ell(A)) \cup \mu_\ell) = \\
&= \sum_{\ell \in S} \text{inv}_\ell ((\omega_\ell + \text{res}_\ell(A)) \cup (\text{res}_\ell(y) + d\mu_\ell)) = \\
&= \sum_{\ell \in S} \text{inv}_\ell ((\omega_\ell + \text{res}_\ell(A)) \cup i_\ell^+(y_\ell^+)) = \\
&= \sum_{\ell \in S} \text{inv}_\ell (\widetilde{\text{res}}_\ell(\omega_\ell + \text{res}_\ell(A)) \cup i_\ell^+(y_\ell^+)) .
\end{aligned}$$

We remark that $\widetilde{\text{res}}_\ell(\omega_\ell + \text{res}_\ell(A))$ and $i_\ell^+(y_\ell^+)$ are cocycles for every ℓ . Furthermore,

$$\text{inv}_\ell (\widetilde{\text{res}}_\ell(\omega_\ell + \text{res}_\ell(A)) \cup i_\ell^+(y_\ell^+)) = 0, \quad \ell \neq p,$$

because $H_f^1(\mathbb{Q}_\ell, V_f) = 0$ for $\ell \neq p$ by the local-global compatibility of the Langlands correspondence. It follows from the definition of $\mathfrak{D}\mathbb{X}_p$ that

$$\widetilde{\text{res}}_p(\omega_p + \text{res}_p(A)) = -\mathfrak{D}\mathbb{X}_p + \widetilde{\text{res}}_p(B)$$

for some $B \in C^1(G_{\mathbb{Q},S}, V_f)$. An easy computation shows that

$$\text{inv}_p(\widetilde{\text{res}}_p(B) \cup i_p^+(y_p)) = \sum_{\ell \in S} \text{inv}_\ell(\widetilde{\text{res}}_\ell(B) \cup i_\ell^+(y_\ell)) = 0$$

(see the proof of [31, Proposition 11.3.15]). We therefore conclude that

$$\mathfrak{H}([x_f], [y_f]) = -\text{inv}_p(\mathfrak{D}\mathbb{X}_p \cup i_p^+(y_p^+))$$

and Theorem 4.13 is proved. \square

Definition 4.17. Let $\mathfrak{X} \in H^1(G_{\mathbb{Q},S}, \overline{V}_f)$.

i) We define $\mathfrak{X}^{\text{cyc}} \in H^1(G_{\mathbb{Q},S}, \overline{V}_f)$, $\mathfrak{X}^{\text{wt}} \in H^1(G_{\mathbb{Q},S}, V_f)$ and $\mathfrak{X}^{\text{c-wt}} \in H^1(G_{\mathbb{Q},S}, V_f(\chi^{1/2}))$ to be the images of \mathfrak{X} under obvious projection maps.

ii) If, in addition, $\text{pr}_0(\mathfrak{X}) \in H_f^1(\mathbb{Q}, V_f)$, we denote by $\mathfrak{d}_{\text{cyc}}\mathfrak{X} \in H^1(\widetilde{\mathbb{D}}_f)$, $\mathfrak{d}_{\text{wt}}\mathfrak{X} \in H^1(\widetilde{\mathbb{D}}_f)$ and $\mathfrak{d}_{\text{c-wt}}\mathfrak{X} \in H^1(\widetilde{\mathbb{D}}_f)$ the Bockstein normalized derivatives of the classes $\mathbb{X} = \mathfrak{X}^{\text{cyc}} \bmod X^2$, $\mathbb{X} = \mathfrak{X}^{\text{wt}} \bmod \varpi_\kappa^2$ and $\mathbb{X} = \mathfrak{X}^{\text{c-wt}} \bmod \varpi_\kappa^2$, respectively.

Proposition 4.18. Suppose we are given a class $\mathfrak{X} \in H^1(G_{\mathbb{Q},S}, \overline{V}_f)$ whose image under the natural map $H^1(G_{\mathbb{Q},S}, \overline{V}_f) \rightarrow H^1(G_{\mathbb{Q},S}, V_f)$ lands in $H_f^1(\mathbb{Q}, V_f)$. Then,

$$\mathfrak{d}_{\text{c-wt}}\mathfrak{X} = \frac{\mathfrak{d}_{\text{cyc}}\mathfrak{X}}{2} + \mathfrak{d}_{\text{wt}}\mathfrak{X}.$$

Proof. Let $\Pi_f : H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^\dagger(V_f)) \rightarrow H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_f)$ and $\text{pr}_{\gamma,\kappa} : H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_f) \rightarrow H^1(\widetilde{\mathbb{D}}_f)$ denote the obvious maps. Under our running hypothesis we may write

$$(4.17) \quad \Pi_f(\mathfrak{X}) = \frac{\gamma - 1}{\log \chi(\gamma)} \cdot \mathfrak{X}_\gamma + \varpi_\kappa \cdot \mathfrak{X}_\kappa$$

for some $\mathfrak{X}_\gamma, \mathfrak{X}_\kappa \in H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_{\mathbf{f}})$. Since

$$\begin{aligned} H^0(\widetilde{\mathbb{D}}_f) &= \ker \left(H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_f) \xrightarrow{[\gamma-1]} H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_f) \right) \\ &= \ker \left(H^1(\widetilde{\mathbb{D}}_{\mathbf{f}}) \xrightarrow{[\varpi_\kappa]} H^1(\widetilde{\mathbb{D}}_{\mathbf{f}}) \right) \end{aligned}$$

we can ensure also that

$$\text{pr}_{\gamma, \kappa}(\mathfrak{X}_\gamma) \otimes 1 = \mathfrak{d}_{\text{cyc}} \mathfrak{X} \quad \text{and} \quad \text{pr}_{\gamma, \kappa}(\mathfrak{X}_\kappa) \otimes 1 = \mathfrak{d}_{\text{wt}} \mathfrak{X}.$$

Let us rewrite (4.17) in the following form:

$$(4.18) \quad \Pi_{\mathbf{f}}(\mathfrak{X}) = \left(\frac{\gamma - 1}{\log \chi(\gamma)} - \frac{\varpi_\kappa}{2} \right) \cdot \mathfrak{X}_\gamma + \varpi_\kappa \cdot \left(\frac{\mathfrak{X}_\gamma}{2} + \mathfrak{X}_\kappa \right).$$

Recall that we have set $\Theta = \frac{\gamma - \chi^{1/2}(\gamma^{-1})}{\log \chi(\gamma)}$ and denote by I_Θ the ideal of \mathcal{H}_A generated by Θ . Then $\mathfrak{X}^{\text{cc}} \in H^1(\mathbb{D}_{\text{rig}}^\dagger(V_{\mathbf{f}}(\chi^{1/2})))$ is the image of \mathfrak{X} under the compositum

$$H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^\dagger(V_{\mathbf{f}})) \xrightarrow{\sim} H^1(\mathbb{D}_{\text{rig}}^\dagger(\overline{V}_{\mathbf{f}})) \longrightarrow H^1(\mathbb{D}_{\text{rig}}^\dagger(V_{\mathbf{f}}(\chi^{1/2}))),$$

where the second map is induced from the projection

$$\text{pr}_\Theta : \overline{V}_{\mathbf{f}} \longrightarrow \overline{V}_{\mathbf{f}}/I_\Theta \overline{V}_{\mathbf{f}} \cong V_{\mathbf{f}}(\chi^{1/2}).$$

that we have discussed in detail as part of Remark 4.6. Multiplication by $\gamma - \chi^{1/2}(\gamma)$ yields an isomorphism

$$[\gamma - \chi^{1/2}(\gamma)] : \overline{V}_{\mathbf{f}} \xrightarrow{\sim} (\gamma^{-1} - \chi^{1/2}(\gamma)) \cdot \overline{V}_{\mathbf{f}} = I_\Theta \cdot \overline{V}_{\mathbf{f}}$$

(where the inversion of γ is due to the fact that it acts on \mathcal{H}_A^ι) that in turn induces a natural inclusion

$$(\gamma - \chi^{1/2}(\gamma)) \cdot H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\mathbf{f}}) \subset H^1(G_p, I_\Theta \overline{V}_{\mathbf{f}}).$$

It therefore follows that

$$(4.19) \quad (\gamma - \chi^{1/2}(\gamma)) \cdot H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\mathbf{f}}) \subset \ker \left(H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\mathbf{f}}) \xrightarrow{\text{pr}_\Theta} H^1(G_p, V_{\mathbf{f}}(\chi^{1/2})) \right).$$

We have a commutative diagram

$$\begin{array}{ccc} H_{\text{Iw}}^1(\mathbb{Q}_p, V_{\mathbf{f}}) & \xrightarrow{\Pi_{\mathbf{f}}} & H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_{\mathbf{f}}) \\ \downarrow \text{pr}_\Theta & & \downarrow \text{pr}_\Theta \\ H^1(G_p, V_{\mathbf{f}}(\chi^{1/2})) & \xrightarrow{\Pi} & H^1(\widetilde{\mathbb{D}}_{\mathbf{f}}(\chi^{1/2})) \end{array}$$

Using (4.18), (4.19) and the Taylor expansion

$$\frac{\gamma - \chi^{1/2}(\gamma)}{2} = \frac{\gamma - 1}{2} - \frac{\varpi_\kappa}{2} + C \varpi_\kappa^2$$

of $(\gamma - \chi^{1/2}(\gamma))/2$ we infer that

$$\Pi(\mathfrak{X}^{\text{cc}}) = \Pi \circ \text{pr}_\Theta(\mathfrak{X}) = \varpi_\kappa \cdot \text{pr}_\Theta \left(\frac{\mathfrak{X}_\gamma}{2} + \mathfrak{X}_\kappa \right) - C \cdot \varpi_\kappa^2 \cdot \text{pr}_\Theta(\mathfrak{X}_\gamma).$$

By the commutativity of the diagram

$$\begin{array}{ccc} H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_{\mathbf{f}}) & \xrightarrow{\text{pr}_{\Theta}} & H^1(\widetilde{\mathbb{D}}_{\mathbf{f}}(\chi^{1/2})) \\ & \searrow \text{pr}_{\kappa, \gamma} & \downarrow \text{pr}_{1/2} \\ & & H^1(\widetilde{\mathbb{D}}_f), \end{array}$$

where the map $\text{pr}_{1/2}$ is induced by the natural reduction map modulo ϖ_{κ} , we see that the element

$$\frac{\mathfrak{d}_{\text{cyc}} \mathfrak{X}}{2} + \mathfrak{d}_{\text{wt}} \mathfrak{X} = \text{pr}_{\kappa, \gamma} \left(\frac{\mathfrak{X}_{\gamma}}{2} + \mathfrak{X}_{\kappa} \right)$$

verifies the condition (i) of Lemma 4.12:

$$i \left(\frac{\mathfrak{X}_{\gamma}}{2} + \mathfrak{X}_{\kappa} \right) = \widetilde{\text{res}}_p(\mathfrak{X}^{\text{cc}}).$$

To conclude with the proof of our proposition, we next check that it satisfies also the condition (ii) of Lemma 4.12. We denote by β^{cyc} , β^{wt} and $\beta^{\text{c-wt}}$ the coboundary maps $\widetilde{H}_f^1(V_f) \rightarrow \widetilde{H}_f^2(V_f) \otimes \mathfrak{I}/\mathfrak{I}^2$ (where $I = J_E$ or \mathfrak{p} , depending on whether we are in the situation of (A) or (B)) associated to the respective families \overline{V}_f , $V_{\mathbf{f}}$ and $V_{\mathbf{f}}(\chi^{1/2})$ of Galois representations. Starting off with the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{p}V_{\mathbf{f}}/\mathfrak{p}^2V_{\mathbf{f}} & \longrightarrow & V_{\mathbf{f}}/\mathfrak{p}^2V_{\mathbf{f}} & \longrightarrow & V_f \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow = \\ 0 & \longrightarrow & \mathfrak{P}\overline{V}_{\mathbf{f}}/\mathfrak{P}^2\overline{V}_{\mathbf{f}} & \longrightarrow & \overline{V}_{\mathbf{f}}/\mathfrak{P}^2\overline{V}_{\mathbf{f}} & \longrightarrow & V_f \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & J_E\overline{V}_f/J_E^2\overline{V}_f & \longrightarrow & \overline{V}_f/J_E^2\overline{V}_f & \longrightarrow & V_f \longrightarrow 0 \end{array}$$

we obtain the following commutative diagram:

$$\begin{array}{ccc} & \widetilde{H}_f^2(V_f) \otimes_E \mathfrak{p}/\mathfrak{p}^2, & \\ \beta^{\text{wt}} \nearrow & \uparrow & \\ \widetilde{H}_f^1(V_f) & \xrightarrow{\beta} \widetilde{H}_f^2(V_f) \otimes_E \mathfrak{P}/\mathfrak{P}^2 & \\ \beta^{\text{cyc}} \searrow & \downarrow & \\ & \widetilde{H}_f^2(V_f) \otimes_E J_E/J_E^2 & \end{array}$$

We therefore have,

$$\beta([x_{\mathbf{f}}]) = \beta^{\text{cyc}}([x_{\mathbf{f}}]) \cdot (\gamma - 1) + \beta^{\text{wt}}([x_{\mathbf{f}}]) \cdot \varpi_{\kappa} = -(\partial_1(\mathfrak{d}_{\text{cyc}}(\mathfrak{X})) \cdot (\gamma - 1) + \partial_1(\mathfrak{d}_{\text{wt}}(\mathfrak{X})) \cdot \varpi_{\kappa})$$

and

$$\beta^{\text{c-wt}}([x_{\mathbf{f}}]) = \beta([x_{\mathbf{f}}]) \Big|_{\gamma-1=\frac{\varpi_{\kappa}}{2}} = - \left(\frac{\mathfrak{d}_{\text{cyc}} \mathfrak{X}}{2} + \mathfrak{d}_{\text{wt}} \mathfrak{X} \right) \cdot \varpi_{\kappa}.$$

The proof of the proposition is now complete. \square

Theorem 4.19 (Central Critical Rubin-style formula for the p -adic height). *Let $[x_f] = [(x, x_\ell, \lambda_\ell)]$ and $[y_f] = [(y, y_\ell, \mu_\ell)] \in \tilde{H}_f^1(V_f)$ be two elements such that $[x] \in H_f^1(\mathbb{Q}, V_f)$. Suppose further that there is an element $\mathfrak{X} \in H^1(G_{\mathbb{Q}, S}, \bar{V}_f)$ with the property that $\text{pr}_0(\mathfrak{X}) = [x]$. Then,*

$$\mathfrak{h}_f^{\text{c-wt}}([x_f], [y_f]) = -\frac{\langle \mathfrak{d}_{\text{cyc}} \mathfrak{X}, [y_p^+] \rangle}{2} - \langle \mathfrak{d}_{\text{wt}} \mathfrak{X}, [y_p^+] \rangle.$$

Proof. This follows from Corollary 4.16 (used in the situation of **(B)** in Section 4.4) and Proposition 4.18. \square

5. THE TWO-VARIABLE PERRIN-RIOU LOGARITHM

5.1. Perrin-Riou's logarithm: General Theory. In this subsection we review the construction of the large exponential map for crystalline (φ, Γ) -modules of rank 1 with coefficients in an affinoid algebra A . We refer the reader to [29] and [28] for general constructions and more results.

Let $\delta : \mathbb{Q}_p^* \rightarrow A^*$ be a continuous character. Recall that we denote by $\mathcal{R}_A(\delta)$ the (φ, Γ) -module $\mathcal{R}_A \cdot e_\delta$ of rank 1 defined by

$$\varphi(e_\delta) = \delta(p) \cdot e_\delta, \quad \tau(e_\delta) = \delta(\chi(\tau)) \cdot e_\delta, \quad \tau \in \Gamma.$$

Set $\alpha(x) = \delta(p) \in A$. We assume that δ is crystalline and of constant slope, namely that

- a) $\delta|_{\mathbb{Z}_p^*}(u) = u^m$ for some integer $m \geq 1$.
- b) The function $x \mapsto v_p(\alpha(x))$ is constant on $U = \text{Spm}(A)$.

The crystalline module $\mathcal{D}_{\text{cris}}(\mathcal{R}_A(\delta))$ associated to $\mathcal{R}_A(\delta)$ is the free A -module of rank 1 generated by $d_\delta = t^{-m}e_\delta$. Note that

$$\varphi(d_\delta) = p^{-m}\delta(p)d_\delta.$$

The Iwasawa cohomology $H_{\text{Iw}}^1(\mathcal{R}_A(\delta))$ is canonically isomorphic to $\mathcal{R}_A(\delta)^{\Delta, \psi=1}$. Set $\mathcal{E}_A = \mathcal{R}_A \cap A[[\pi]]$. The set $\mathcal{E}_A^{\Delta, \psi=0}$ is the free \mathcal{H}_A -submodule of \mathcal{E}_A generated by

$$1 + \pi_0 = 1 + \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \sigma(\pi).$$

We equip \mathcal{E}_A with the operator $\partial = (1 + \pi) \frac{d}{d\pi}$.

Let $z \in \mathcal{D}_{\text{cris}}(\mathcal{R}_A(\delta)) \otimes_A \mathcal{E}_A^{\Delta, \psi=0}$. It may be shown that the equation

$$(\varphi - 1)F = z - \frac{\partial^m z(0)}{m!} t^m, \quad t = \log(1 + \pi)$$

has a solution in $\mathcal{D}_{\text{cris}}(\mathcal{R}_A(\delta)) \otimes_A \mathcal{E}_A$ and we define

$$\text{Exp}_{\mathcal{R}(\delta)}^\varepsilon(z) = (-1)^m \frac{\log \chi(\gamma)}{p} t^m \partial^m(F).$$

Exactly as in the classical case $A = E$ (see [9]), it is not hard to check that $\text{Exp}_{\mathcal{R}(\delta)}^\varepsilon(z) \in \mathcal{R}(\delta)^{\Delta, \psi=1}$. Therefore we have a well defined map

$$\text{Exp}_{\mathcal{R}(\delta)}^\varepsilon : \mathcal{D}_{\text{cris}}(\mathcal{R}_A(\delta)) \otimes_A \mathcal{E}_A^{\Delta, \psi=0} \rightarrow H_{\text{Iw}}^1(\mathcal{R}_A(\delta)).$$

Now let V be a p -adic representation with coefficients in A and let $\mathbb{D} = \mathcal{R}_A(\delta)$ be a crystalline (φ, Γ) -submodule of $\mathbb{D}_{\text{rig}, A}^\dagger(V)$ of constant slope. We denote by

$$\text{Exp}_{V, \mathbb{D}}^\varepsilon : \mathcal{D}_{\text{cris}}(\mathbb{D}) \otimes_A \mathcal{E}_A^{\Delta, \psi=0} \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, V)$$

the composition of $\text{Exp}_{\mathcal{R}(\delta)}^\varepsilon$ with the canonical map $H_{\text{Iw}}^1(\mathcal{R}_A(\delta)) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, V)$.

The cohomological pairings

$$\langle \cdot, \cdot \rangle_n : H^1(K_n, V^*(1)) \times H^1(K_n, V) \rightarrow A$$

give rise to a Λ_A -linear pairing

$$\langle \cdot, \cdot \rangle_{\text{Iw}} : H_{\text{Iw}}^1(\mathbb{Q}_p, V^*(1)) \times H_{\text{Iw}}^1(\mathbb{Q}_p, V)^\iota \rightarrow \Lambda_A.$$

By linearity, this pairing extends to

$$\langle \cdot, \cdot \rangle_{\text{Iw}} : H_{\text{Iw}}^1(\mathbb{Q}_p, V^*(1)) \otimes_{\Lambda_A} \mathcal{H}_A \times H_{\text{Iw}}^1(\mathbb{Q}_p, V)^\iota \otimes_{\Lambda_A} \mathcal{H}_A \rightarrow \mathcal{H}_A.$$

For any $\eta \in \mathcal{D}_{\text{cris}}(\mathbb{D})$ the element $\tilde{\eta} = \eta \otimes (1 + \pi_0)$ lies in $\mathcal{D}_{\text{cris}}(\mathbb{D}) \otimes \mathcal{E}_A^{\Delta, \psi=0}$ and we define a map

$$\mathfrak{Log}_{V^*(1), \eta}^\varepsilon : H_{\text{Iw}}^1(\mathbb{Q}_p, V^*(1)) \rightarrow \mathcal{H}_A$$

by

$$\mathfrak{Log}_{V^*(1), \eta}^\varepsilon(z) = \left\langle z, \text{Exp}_{V, \mathbb{D}}^{\varepsilon^{-1}}(\tilde{\eta})^\iota \right\rangle_{\text{Iw}}.$$

The following theorem summarizes the main properties of these maps.

Theorem 5.1. i) The maps $\text{Exp}_{V, \mathbb{D}}^\varepsilon$ and $\mathfrak{Log}_{V^*(1), \eta}^\varepsilon$ commute with the base change.

ii) Assume that $x \in \text{Spm}(A)$ is an E -valued point such that V_x is semistable at x . Then $\text{Exp}_{V_x, \mathbb{D}_x}^\varepsilon$ coincides with the restriction of Perrin-Riou's large exponential map

$$\text{Exp}_{V_x, m}^\varepsilon : \mathbb{D}_{\text{cris}}(V_x) \otimes_A \mathcal{E}_A^{\Delta, \psi=0} \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, V_x)$$

on $\mathcal{D}_{\text{cris}}(\mathbb{D}_x) \otimes_A \mathcal{E}_A^{\Delta, \psi=0}$. Therefore, $\mathfrak{Log}_{V_x^*(1), \eta_x}^\varepsilon$ coincides with Perrin-Riou's large logarithm map as defined in [4].

iii) Let $\mathfrak{a} : \mathcal{H}_A \rightarrow A$ denote the augmentation map induced from $\gamma \mapsto 1$. Then

$$\mathfrak{a} \circ \mathfrak{Log}_{V^*(1), \eta}^\varepsilon(z) = (m-1)! \frac{1 - p^{-1}\alpha(x)^{-1}}{1 - \alpha(x)} \text{Log}_{V^*(1), \eta}(\text{pr}_0(z)),$$

where $\text{Log}_{V^*(1), \eta}(\text{pr}_0(z)) = \langle \text{pr}_0(z), \exp_{\mathbb{D}}(\eta) \rangle \in A$ and $\exp_{\mathbb{D}}$ is the Bloch-Kato exponential map (for its general definition, see [29]).

Proof. i) This follows directly from constructions.

ii) This portion follows from i) and Berger's interpretation in [9] of the large exponential map; more precisely, by Berger's result that Perrin-Riou's large exponential map can be defined via (5.1).

iii) By the construction of the large exponential map we have

$$\text{pr}_0(\text{Exp}_{V, \mathbb{D}}^{\varepsilon^{-1}}(\tilde{\eta})^\iota) = (m-1)! \exp_V \left(\frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \eta \right)$$

(see [34] or [7, Corollaire 4.10], for the proof in the case $A = E$. The same computation proves this formula in the general case.) Therefore

$$\mathfrak{a} \circ \mathfrak{Log}_{V^*(1), \eta}^\varepsilon(z) = (m-1)! \frac{1 - p^{-1}\alpha(x)^{-1}}{1 - \alpha(x)} \langle \text{pr}_0(z), \exp_V(\eta) \rangle.$$

□

5.2. Perrin-Riou's logarithm for modular forms. We apply the theory outlined in the previous section to the situation studied in Section 4.2. Let (V_f, \mathbb{D}_f) be the triangulation associated to a p -adic family \mathbf{f} of cuspidal eigenforms passing through f and an eigenvalue α of f . We fix ε and write Exp_f for the sake of brevity to denote the exponential map

$$\text{Exp}_{V_f, \mathbb{D}}^\varepsilon : \mathbb{D}_{\text{cris}}(V_f)^{\varphi=\alpha} \otimes_E \mathcal{E}^{\Delta, \psi=0} \longrightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, V_f).$$

Recall that we denote by d_δ the element $e_\delta \otimes t^{-k/2}$. Set $\eta = d_\delta$. Using the skew-symmetric pairing $V_f \times V_f \rightarrow E$, we may consider the map $\mathfrak{Log}_{V_f, \eta}^\varepsilon$. In order to simplify the notation, we shall denote this map by \mathfrak{Log}_f .

Assume that $a_p = p^{k/2-1}$. Then $\alpha = \alpha(k) = p^{-1}$ and from Theorem 5.1 it follows that $\text{Exp}_f(d_\delta \otimes (1 + \pi_0)) \in (\gamma - 1) \cdot \mathcal{H}_E$. Let pr_0 denote the natural projection $H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^\dagger(V_f)) \rightarrow H^1(G_p, V_f)$.

Proposition 5.2. i) *There exists a unique $F \in H_{\text{Iw}}^1(\mathbb{D}_f)$ such that*

$$(\gamma - 1)F = \text{Exp}_f(d_\delta \otimes (1 + \pi_0)).$$

ii) *For every $z \in H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^\dagger(V_f))$ we have*

$$\left(1 - \frac{1}{p}\right) \langle z, F \rangle_{\text{Iw}} \equiv \left(\frac{k}{2} - 1\right)! (\log \chi(\gamma))^{-1} \langle \Pi_f \circ \text{pr}_\gamma(z), \Psi_1^* \rangle \pmod{J_E^2},$$

where $\Pi_f : H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^\dagger(V_f)) \rightarrow H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_f)$ and $\text{pr}_\gamma : H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_f) \rightarrow H^1(\widetilde{\mathbb{D}}_f)$ denote the natural projections. In particular,

$$\left\langle z, \text{Exp}_f^{\varepsilon^{-1}}((1 + X) \otimes d_\delta) \right\rangle_{\text{Iw}} \in J_E^2 \iff \text{pr}_0(z) \in H_f^1(\mathbb{Q}_p, V).$$

iii) *Assume in addition that $\text{pr}_0(z) \notin H_f^1(G_p, V_f)$ and define*

$$L_p(z, s) = \mathcal{A}(\mathfrak{Log}_f(z))(-s) = \mathfrak{Log}_f(z)(\chi(\gamma)^s - 1).$$

Then,

$$L'_p(z, s) = -\langle z, F \rangle \equiv \Gamma(k/2 - 1) \left(1 - \frac{1}{p}\right)^{-1} \mathcal{L}_{\text{FM}}(f) \left[d_\delta, \exp_{V_f}^*(\text{pr}_0(z)) \right]_{V_f} \cdot s \pmod{s^2},$$

where $[\cdot, \cdot]_{V_f} : \mathbb{D}_{\text{st}}(V_f) \times \mathbb{D}_{\text{st}}(V_f) \rightarrow E$ denotes the canonical duality.

Proof. i) and ii) are proved in [4, Proposition 1.3.7]; whereas the assertion iii) is proved in [4, Proposition 2.2.2]. □

Recall that we have a duality

$$V_{\mathbf{f}}(\chi) \times V_{\mathbf{f}} \rightarrow E(\chi).$$

We denote by $\mathfrak{Log}_{\mathbf{f}}$ the large logarithm map $\mathfrak{Log}_{V_{\mathbf{f}}(\chi), \eta}^\varepsilon$ for $\eta = d_\delta$. Let

$$\mathbf{z} \in H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}, A}^\dagger(V_{\mathbf{f}}(\chi))).$$

We define

$$L_p(\mathbf{z}, \kappa, s) := \mathcal{A}(\mathfrak{Log}_{\mathbf{f}}(\mathbf{z}))(\kappa, -s) = (\mathfrak{Log}_{\mathbf{f}}(\mathbf{z}))((1 + p)^{\kappa-k} - 1, \chi(\gamma)^s - 1).$$

Note that $L_p(\mathbf{z}, \kappa, s)$ is a locally analytic function in the weight variable κ and the cyclotomic variable s . Write $\mathbf{z}(k) \in H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^\dagger(V_f))$ for the stalk of \mathbf{z} at the point $(f, a_p(f))$ (corresponding to $\kappa = k$). We denote by $\Pi_{\mathbf{f}} : H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}, A}^\dagger(V_{\mathbf{f}})) \rightarrow H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_{\mathbf{f}})$ and $\text{pr}_{\gamma, \kappa} : H_{\text{Iw}}^1(\widetilde{\mathbb{D}}_{\mathbf{f}}) \rightarrow H^1(\widetilde{\mathbb{D}}_f)$ the natural projections. Recall that $\mathfrak{J} \subset E[[\kappa - k, s]]$ denotes the ideal generated by $(\kappa - k)$ and s .

Theorem 5.3. (i) Suppose $\mathbf{z} \in H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}, A}^\dagger(V_{\mathbf{f}}))$. Then we have the following equality inside $\mathfrak{J}/\mathfrak{J}^2$:

$$\left(1 - \frac{1}{p}\right) \Gamma(k/2-1)^{-1} L_p(\mathbf{z}, \kappa, s) \equiv \langle \text{pr}_{\gamma, \kappa} \circ \Pi_{\mathbf{f}}(\mathbf{z}), \Psi_1^* \rangle \cdot s - \frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot \langle \text{pr}_{\gamma, \kappa} \circ \Pi_{\mathbf{f}}(\mathbf{z}), \exp_{\mathbb{D}_f}(d_\delta) \rangle \cdot (\kappa - k).$$

(ii) Suppose in addition that $\text{pr}_{\gamma, \kappa} \circ \Pi_{\mathbf{f}}(\mathbf{z}) \neq 0$. Then,

$$\left(1 - \frac{1}{p}\right) \Gamma\left(\frac{k}{2} - 1\right)^{-1} L_p(\mathbf{z}, \kappa, s) \equiv \left(s - \frac{\kappa - k}{2}\right) \mathcal{L}_{\text{FM}}(f)[d_\delta, \exp_V^*(\text{pr}_0(\mathbf{z}(k)))]_{V_f} \pmod{\mathfrak{J}^2}.$$

Proof. i) By Theorem 4.7 together with the fact that $\alpha(k) = p^{-1}$ we have

$$(5.1) \quad 1 - p^{-1} \alpha(\kappa)^{-1} \equiv -\frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot (\kappa - k) \pmod{\mathfrak{J}^2}.$$

Theorem 5.1(iii) together with Proposition 5.2(ii) give

$$\begin{aligned} \mathcal{A} \circ \mathfrak{Log}_{\mathbf{f}}(\mathbf{z}) &\equiv \left(1 - \frac{1}{p}\right)^{-1} \Gamma(k/2 - 1) \langle \text{pr}_{\gamma, \kappa} \circ \Pi_{\mathbf{f}}(\mathbf{z}), \Psi_1^* \rangle \cdot s \\ &\quad - \frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot \left(1 - \frac{1}{p}\right)^{-1} \Gamma(k/2 - 1) \cdot \langle \text{pr}_{\gamma, \kappa} \circ \Pi_{\mathbf{f}}(\mathbf{z}), \exp_{\mathbb{D}_f}(d_\delta) \rangle \cdot (\kappa - k). \end{aligned}$$

ii) This follows from the first part, combined with Proposition 5.2(iii). \square

6. A p -ADIC BEILINSON FORMULA IN TWO-VARIABLES

We keep previous notation and conventions. Suppose we are given a class $\mathfrak{X} \in H^1(G_{\mathbb{Q}, S}, \overline{V}_{\mathbf{f}})$ whose image under the natural map $H^1(G_{\mathbb{Q}, S}, \overline{V}_{\mathbf{f}}) \rightarrow H^1(G_{\mathbb{Q}, S}, V_f)$ lands in $H_{\mathbf{f}}^1(\mathbb{Q}, V_f)$. We denote by $\mathfrak{d}_?(\mathfrak{X})$ the Bockstein normalized partial derivatives for $? \in \{\text{cyc}, \text{wt}, \text{c-wt}\}$. To simplify notation, we write $L_p(\mathfrak{X}, \kappa, s)$ for the p -adic L -function associated to $\text{res}_p(\mathfrak{X}) \in H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}, A}^\dagger(V_{\mathbf{f}}))$.

Proposition 6.1. We have the following equality inside $\mathfrak{J}^2/\mathfrak{J}^3$:

$$\begin{aligned} \left(1 - \frac{1}{p}\right) \Gamma(k/2-1)^{-1} L_p(\mathfrak{X}, \kappa, s) &= \\ &\langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), \Psi_1^* \rangle \cdot s^2 - \frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot \langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \exp(d_\alpha) \rangle \cdot (\kappa - k)^2 \\ &+ \left(\langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \Psi_1^* \rangle - \frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot \langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), \exp(d_\alpha) \rangle \right) \cdot s(\kappa - k). \end{aligned}$$

In particular, all the three quantities $\langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), \Psi_1^* \rangle$, $\langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \exp(d_\alpha) \rangle$ and $\left(\langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \Psi_1^* \rangle - \frac{\mathcal{L}_{\text{FM}}(f)}{2} \cdot \langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), \exp(d_\alpha) \rangle \right)$ are independent of choices involved in the definitions of $\mathfrak{d}_{\text{cyc}}(\mathfrak{X})$ and $\mathfrak{d}_{\text{wt}}(\mathfrak{X})$.

Proof of Proposition 6.1. It follows from the A -linearity of the large regulator map \mathfrak{Log}_f that

$$\mathfrak{Log}_f(\mathfrak{X}) = \frac{\gamma - 1}{\log_p \chi(\gamma)} \cdot \mathfrak{Log}_f(\mathfrak{X}^{(\gamma)}) + \varpi_\kappa \cdot \mathfrak{Log}_f(\mathfrak{X}^{(\kappa)})$$

and the proof follows applying the Amice transform on both sides and using Theorem 5.3(i) (with $\mathbf{z} = \mathfrak{X}^{(\gamma)}$ and $\mathbf{z} = \mathfrak{X}^{(\kappa)}$). \square

Now we can prove one the the main results of this paper.

Theorem 6.2. *Let $\mathfrak{X} \in H^1(G_{\mathbb{Q},S}, \overline{V}_f)$ and let $[x] = \text{pr}_0(\mathfrak{X}) \in H^1(G_{\mathbb{Q},S}, V_f)$ denote its canonical projection.*

i) *Inside $\mathfrak{I}/\mathfrak{I}^2$ we have the following equality:*

$$\begin{aligned} \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \left(1 - \frac{1}{p} \right) \Gamma(k/2 - 1)^{-1} L_p(\mathfrak{X}, \kappa, s) \\ = \mathcal{L}_{\text{FM}}(f) \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle [d_\delta, \exp_V^*(\text{res}_p([x]))]_{V_f} \cdot \left(s - \frac{\kappa - k}{2} \right) \\ = [d_\delta, \exp_V^*(\text{res}_p([x]))]_{V_f} \cdot \mathcal{A}(\mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), \partial_0(d_{\tilde{\delta}}))). \end{aligned}$$

In particular, $L_p(\mathfrak{X}, \kappa, s) \in \mathfrak{I}^2$ if and only if $[x] \in H_f^1(\mathbb{Q}, V_f)$.

ii) *Suppose that $[x] \in H_f^1(V_f)$ and denote by $[x_f] \in \tilde{H}_f^1(V_f)$ the canonical lift of $[x]$.*

$$\left(1 - \frac{1}{p} \right) \Gamma\left(\frac{k}{2} - 1\right)^{-1} \langle \Psi_2, \text{res}_p([x]) \rangle \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle \cdot L_p(\mathfrak{X}, \kappa, s) = \mathcal{A}(\mathfrak{Reg}_{\mathbb{H}_f}(\partial_0(d_{\tilde{\delta}}), [x_f]))$$

Proof. In (i), the first asserted equality follows from Theorem 5.3(ii) (on eliminating the redundant $\langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle$ factors from both sides using our running hypothesis that $\mathcal{L}_{\text{FM}}(f) \neq 0$) and the second equality from (4.10).

The proof of (ii) is rather a tedious computation. To ease notation, we set $C = \left(1 - \frac{1}{p} \right) \Gamma\left(\frac{k}{2} - 1\right)^{-1}$. We also write \mathcal{L} in place of $\mathcal{L}_{\text{FM}}(f)$. Using (4.10), (4.9) together with Theorem 4.10(i) and Corollary 4.16 we have,

$$\begin{aligned} \mathcal{A}(\mathfrak{Reg}_{\mathbb{H}_f}(\partial_0(d_{\tilde{\delta}}), [x_f])) &= \mathcal{A} \circ \det \begin{pmatrix} \mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), \partial_0(d_{\tilde{\delta}})) & \mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), [x_f]) \\ \mathbb{H}_f([x_f], \partial_0(d_{\tilde{\delta}})) & \mathbb{H}_f([x_f], [x_f]) \end{pmatrix} \\ &= \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \cdot \langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), [x_p^+] \rangle \cdot \left(s - \frac{\kappa - k}{2} \right)^2 \\ &\quad + \langle \Psi_2, [x_p^+] \rangle^2 \cdot s \cdot (\kappa - k - s), \end{aligned}$$

where we used the identity (that follows from the functional equation for the two-variable height pairing)

$$(6.1) \quad \begin{aligned} \mathcal{A}(\mathbb{H}_f([x_f], \partial_0(d_{\tilde{\delta}}))) (\kappa, s) &= -\mathcal{A}(\mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), [x_f])) (\kappa, \kappa - s - k) \\ &= -\langle \Psi_2, [x_p^+] \rangle \cdot (\kappa - k - s). \end{aligned}$$

We define $\Phi_1 := \Psi_1$ and $\Phi_2 := \Psi_2 + \mathcal{L} \cdot \Psi_1$. These two elements constitute a basis of $H^1(\widetilde{\mathbb{D}}_f)$ and enjoy the properties that

$$\langle \Phi_2, \partial_0(d_{\tilde{\delta}}) \rangle = 0 = \langle \Phi_1, [x_p^+] \rangle$$

where the second equality follows from Corollary 4.4. Write

$$\mathfrak{d}_{\text{cyc}}(\mathfrak{X}) = a_1(\gamma)\Phi_1 + a_2(\gamma)\Phi_2, \quad \mathfrak{d}_{\text{wt}}(\mathfrak{X}) = a_1(\kappa)\Phi_1 + a_2(\kappa)\Phi_2.$$

We have,

$$(6.2) \quad \begin{aligned} \mathcal{A}(\mathfrak{Reg}_{\mathbb{H}_f}(\partial_0(d_{\tilde{\delta}}), [x_f])) &= \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \cdot a_2(\gamma) \langle \Psi_2, [x_p^+] \rangle \cdot \left(s - \frac{\kappa - k}{2}\right)^2 \\ &\quad + \langle \Psi_2, [x_p^+] \rangle^2 \cdot s \cdot (\kappa - k - s) \end{aligned}$$

Set $C_1 = \langle \Psi_2, [x_p^+] \rangle$ and $C_2 = C_1 \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle$. Note that C_2 is zero iff C_1 is. We have the following identities.

- The equation (4.9) and the Rubin-style formula (Corollary 4.16) for the cyclotomic height pairing \mathfrak{h}_p together show that

$$(6.3) \quad \langle \Psi_2, [x_p^+] \rangle = a_1(\gamma) \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle.$$

- We have

$$(6.4) \quad \begin{aligned} \mathfrak{h}_f^{\text{c-wt}}([x_f], \partial_0(d_{\tilde{\delta}})) &= -\langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \partial_0(d_{\tilde{\delta}}) \rangle - \frac{1}{2} \langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), \partial_0(d_{\tilde{\delta}}) \rangle \\ &= -a_1(\kappa) \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle + \frac{\mathfrak{h}_p([x_f], \partial_0(d_{\tilde{\delta}}))}{2} \end{aligned}$$

where we used Theorem 4.19 for the first equality and the Rubin-style formula for the cyclotomic height-pairing \mathfrak{h}_p for the second. Furthermore, it follows from (4.9) and (4.11) that

$$\mathfrak{h}_f^{\text{c-wt}}([x_f], \partial_0(d_{\tilde{\delta}})) = -\frac{\mathfrak{h}_p([x_f], \partial_0(d_{\tilde{\delta}}))}{2} = \frac{\langle \Psi_2, [x_p^+] \rangle}{2}$$

This combined with (6.4) yields

$$(6.5) \quad \begin{aligned} a_1(\kappa) &= -\frac{\langle \Psi_2, [x_p^+] \rangle}{\langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle} \\ &= -a_1(\gamma). \end{aligned}$$

- Using Theorem 4.19 along with the fact that the pairing $\mathfrak{h}_f^{\text{c-wt}}$ is skew-symmetric, it follows that

$$\begin{aligned} 0 = \mathfrak{h}_f^{\text{c-wt}}([x_f], [x_f]) &= -\langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), [x_p^+] \rangle - \frac{1}{2} \langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), [x_p^+] \rangle \\ &= -a_2(\kappa) \cdot \langle \Psi_2, [x_p^+] \rangle - a_2(\gamma) \cdot \frac{\langle \Psi_2, [x_p^+] \rangle}{2}, \end{aligned}$$

where we used the fact that $\langle \Phi_1, [x_p^+] \rangle = 0$ for the second line. We conclude that

$$(6.6) \quad \langle \Psi_2, [x_p^+] \rangle \cdot a_2(\kappa) = -\frac{\langle \Psi_2, [x_p^+] \rangle}{2} \cdot a_2(\gamma).$$

We compute the coefficients in the expression for $\mathcal{A}(\mathfrak{Reg}_{\mathbb{H}_f}(\partial_0(d_{\tilde{\delta}}), [x_f]))$.

(1) *The coefficient A_s of s^2 .* This coefficient equals

$$\begin{aligned} A_s &= \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \cdot a_2(\gamma) \langle \Psi_2, [x_p^+] \rangle - \langle \Psi_2, [x_p^+] \rangle^2 \\ &= C_1 \cdot (a_2(\gamma) \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle - \langle \Psi_2, [x_p^+] \rangle) \\ &= C_1 \cdot (-\mathcal{L} \cdot a_2(\gamma) \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle - \langle \Psi_2, [x_p^+] \rangle) \\ &= C_1 \cdot (-\mathcal{L} \cdot a_2(\gamma) \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle - a_1(\gamma) \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle) \\ &= -C_2 (\mathcal{L} \cdot a_2(\gamma) + a_1(\gamma)) = C_2 \cdot \langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), \Psi_1^* \rangle. \end{aligned}$$

Here the third equality is because $\langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle = -\mathcal{L} \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle$, the fourth follows from (6.3). This agrees with the coefficient of s^2 in the degree Taylor polynomial for

$$\left(1 - \frac{1}{p}\right) \Gamma\left(\frac{k}{2} - 1\right)^{-1} \langle \Psi_2, \text{res}_p([x]) \rangle \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle \cdot L_p(\mathfrak{X}, \kappa, s).$$

(2) *The coefficient $A_{s,\kappa}$ of $s \cdot (\kappa - k)$ in the expression for $\mathcal{A}(\mathfrak{Reg}_{\mathbb{H}_f}(\partial_0(d_{\tilde{\delta}}), [x_f]))$.* This coefficient equals,

$$\begin{aligned} A_{s,\kappa} &= C_1 (\langle \Psi_2, [x_p^+] \rangle - \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \cdot a_2(\gamma)) \\ &= C_2 (\mathcal{L} \cdot a_2(\gamma) + a_1(\gamma)) \\ &= C_2 \left(-\mathcal{L} \cdot a_2(\kappa) + \frac{\mathcal{L}}{2} \cdot a_2(\gamma) - a_1(\kappa) \right) \\ &= C_2 \left(\langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \Psi_1^* \rangle + \frac{\mathcal{L}}{2} \cdot \langle \mathfrak{d}_{\text{cyc}}(\mathfrak{X}), \Psi_2^* \rangle \right) \\ &= C_2 \left(\langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \Psi_1^* \rangle - \frac{\mathcal{L}}{2} \langle \exp_{\mathbb{D}_f}(d_\alpha), \mathfrak{d}_{\text{cyc}}(\mathfrak{X}) \rangle \right) \end{aligned}$$

where we used (6.3) for the second equality, (6.5) and (6.6) for the third and [4, Proposition 1.2.6] for the last.

(3) *The coefficient A_κ of $(\kappa - k)^2$ in the expression for $\mathcal{A}(\mathfrak{Reg}_{\mathbb{H}_f}(\partial_0(d_{\tilde{\delta}}), [x_f]))$.* This coefficient equals,

$$\begin{aligned} A_\kappa &= -\frac{1}{4} \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle \cdot a_2(\gamma) \langle \Psi_2, [x_p^+] \rangle \\ &= C_2 \cdot \frac{\mathcal{L}}{2} \cdot a_2(\kappa) \\ &= C_2 \cdot \frac{\mathcal{L}}{2} \cdot \langle \mathfrak{d}_{\text{wt}}(\mathfrak{X}), \Psi_2^* \rangle \\ &= -C_2 \cdot \frac{\mathcal{L}}{2} \cdot \langle \exp_{\mathbb{D}}(d_\alpha), \mathfrak{d}_{\text{wt}}(\mathfrak{X}) \rangle \end{aligned}$$

where we used (6.6) for the second equality. This agrees with the coefficient of $(\kappa - k)^2$ in the degree-2 Taylor polynomial for

$$\left(1 - \frac{1}{p}\right) \Gamma\left(\frac{k}{2} - 1\right)^{-1} \langle \Psi_2, \text{res}_p([x]) \rangle \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle \cdot L_p(\mathfrak{X}, \kappa, s).$$

The proof of the theorem now follows using the computations in the paragraphs (1), (2) and (3) and Proposition 6.1. \square

Corollary 6.3 (Central critical Rubin-style formula for p -adic L -functions). *In the situation of Theorem 6.2, define*

$$\mathfrak{L}_p(\mathfrak{X}, \kappa) = L_p\left(\mathfrak{X}, \kappa, \frac{\kappa - k}{2}\right).$$

Assume that $[x] \in H_{\mathbf{f}}^1(G_{\mathbb{Q}, S}, V_f)$. Then

$$(6.7) \quad \left(1 - \frac{1}{p}\right) \Gamma\left(\frac{k}{2} - 1\right)^{-1} \langle \Psi_2, [x_p^+] \rangle \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle \cdot \frac{d^2}{d\kappa^2} \mathfrak{L}_p(\mathfrak{X}, \kappa) \Big|_{\kappa=k} = \frac{\langle \Psi_2, [x_p^+] \rangle^2}{2}.$$

Proof. This follows from Theorem 6.2(ii) (specialized to $s = \frac{\kappa - k}{2}$), more particularly, see (6.2) as part of its proof. \square

Proposition 6.4. *Suppose that we are in the situation of Corollary 6.3. Then,*

$$\text{ord}_{\kappa=k} \mathfrak{L}_p(\mathfrak{X}, \kappa) > 2 \iff \text{res}_p([x]) = 0.$$

Proof. (\implies) The assumption forces the right hand side of (6.7) to vanish and it follows that $\text{res}_p([x]) = 0$.

(\impliedby) The hypothesis that $\text{res}_p([x]) = 0$ shows that $\text{res}_p(\mathfrak{X}) \in \mathfrak{P} \cdot H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^{\dagger}(V_{\mathbf{f}}))$.

We may therefore write

$$\text{res}_p(\mathfrak{X}) = \frac{\gamma - 1}{\log_p \chi(\gamma)} \cdot \mathbf{z}_{\gamma} + \varpi_{\kappa} \cdot \mathbf{z}_{\kappa}$$

for some $\mathbf{z}_{\gamma}, \mathbf{z}_{\kappa} \in H_{\text{Iw}}^1(\mathbb{D}_{\text{rig}}^{\dagger}(V_{\mathbf{f}}))$ and set

$$z_{\gamma} := \text{pr}_0(\mathbf{z}_{\gamma}), \quad z_{\kappa} := \text{pr}_0(\mathbf{z}_{\kappa}) \in H^1(\mathbb{D}_{\text{rig}}^{\dagger}(V_f)).$$

We have

$$\begin{aligned} \mathfrak{L}_p(\mathfrak{X}, \kappa) &= L_p(\mathfrak{X}, \kappa, s) \Big|_{s=\frac{\kappa-k}{2}} \\ &= ((s-1) \cdot L_p(\mathbf{z}_{\gamma}, \kappa, s) + (\kappa - k) \cdot L_p(\mathbf{z}_{\kappa}, \kappa, s)) \Big|_{s=\frac{\kappa-k}{2}}. \end{aligned}$$

First assume that $z_{\gamma} \in H_{\mathbf{f}}^1(G_p, V_f)$, where $\gamma \in \{\gamma, \kappa\}$. Then the projection of z_{γ} to $H^1(\widetilde{\mathbb{D}})$ is 0 and from Theorem 5.3 (i) it follows that

$$(6.8) \quad L_p(\mathbf{z}_{\gamma}, \kappa, s) \in \mathfrak{I}^2.$$

Now assume that $z_{\gamma} \notin H_{\mathbf{f}}^1(G_p, V_f)$. Write $C = \left(1 - \frac{1}{p}\right) \Gamma\left(\frac{k}{2} - 1\right)^{-1}$ for simplicity. From Theorem 5.3 (ii) we have

$$L_p(\mathbf{z}_{\gamma}, \kappa, s) \equiv \left(s - \frac{\kappa - k}{2}\right) \mathcal{L}_{\text{FM}}(f) \cdot [d_{\delta}, \exp_V^*(z_{\gamma})]_{V_f} \pmod{\mathfrak{I}^2}$$

Taking $s = \frac{\kappa-k}{2}$, we obtain that $\mathfrak{L}(\mathbf{z}_?, \kappa) \equiv 0 \pmod{(\kappa-k)^2}$. Together with (6.8), this implies that $\text{ord}_{\kappa=k} \mathfrak{L}_p(\mathfrak{X}, \kappa) > 2$. \square

7. MAIN RESULTS

7.1. *p*-adic *L*-functions. Let

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

be a elliptic newform of even weight k and level N as before. Let $\varepsilon_{f,N} \in \{\pm 1\}$ denote the eigenvalue of the Fricke involution acting on f and set $\varepsilon_f = (-1)^{k/2} \varepsilon_{f,N}$. According to [39, Theorem 3.66] the sign of ε_f agrees with the sign of the functional equation of the Hecke L -function $L(f, s)$. We suppose throughout that we have $a_p = p^{\frac{k}{2}-1}$. Recall the p -adic L -function $L_{p,\alpha}(f, \omega^{k/2}, s)$ (where ω is the Teichmüller character) of Amice–Vélu, Manin and Višik. We refer the reader to [4, Section 4.1] for the precise definition and basic properties of this function.

Let \mathbf{f} be a family of modular forms with coefficients in $A = \mathcal{O}(U)$ passing through f , where $U = \overline{B}(k, p^{-r})$. Let ϵ be a Dirichlet character of order dividing 2; its choice will be made precise³ below. We shall consider the two-variable Mazur–Kitagawa p -adic L -function $L_p(\mathbf{f}, \epsilon, \kappa, s)$ of Seveso (introduced in [37, Section 5]) associated with the family \mathbf{f} , for s in the neighborhood $k/2 + p\mathbb{Z}_p$ of $k/2$. See also [1, 32] for constructions of related p -adic L -functions and Remark 7.1 for a comparison of Seveso’s notation with that of [4, 27] (according to which our normalizations are made). At each classical point x of weight $k_x \in \mathbb{Z}_{\geq 2} \cap \overline{B}(k, p^{-r})$ we have

$$(7.1) \quad L_p(\mathbf{f}, \epsilon, k_x, s) = \lambda(x) L_p(\mathbf{f}_x, \epsilon, s),$$

where $\lambda(x)$ is Stevens’s interpolation factor; see [11, Theorem 1.5] for its defining property and [37, Theorem 5.3]. Here $L_p(\mathbf{f}_x, \epsilon, s)$ is in Seveso’s notation and its comparison to the p -adic L -function denoted by $L_{p,\alpha(x)}(f_x, \epsilon \omega^{k/2}, s)$ in [4] is included below.

Remark 7.1. Given a classical point x of even weight $k_x \in \mathbb{Z}_{\geq 2}$ as above, the p -adic L -function $L_p(\mathbf{f}_x, \epsilon \omega^{k/2}, s)$ of Manin–Višik and Amice–Velu which is denoted in [4] by $L_{p,\alpha(x)}(\mathbf{f}_x, \epsilon \omega^{k/2}, s)$ and corresponds to the function denoted in [27] by the notation $L_p(\mathbf{f}_x, \alpha(x), \epsilon \omega^{k/2-1}, s-1)$. We also note that

$$\begin{aligned} L_p(\mathbf{f}_x, \alpha(x), \epsilon \omega^{k/2-1}, s-1) &= L_p(\mathbf{f}_x, \alpha(x), \epsilon \omega^{k/2-1} \chi_{s-1}) \\ &= L_p(\mathbf{f}_x, \alpha(x), \epsilon \omega^{k/2-s} \chi^{s-1}) \end{aligned}$$

(still in the notation of [27], so that χ_s is as in Section 13.2 of loc. cit.) where we make sense of the second line only for integer values of s .

Let ψ denote an arbitrary primitive Dirichlet character. In the notation of [37], the function $L_p(\mathbf{f}_x, \psi, s)$ in the neighborhood $j + p\mathbb{Z}_p$ of j corresponds to the p -adic L -function

$$\begin{aligned} L_p(\mathbf{f}_x, \alpha(x), \psi \chi^{s-1}) &= L_p(\mathbf{f}_x, \alpha(x), \psi \omega^{j-1} \chi_{s-1}) \\ &= L_p(\mathbf{f}_x, \alpha(x), \psi \omega^{j-1}, s-1) \end{aligned}$$

of [27]. (Note that Seveso has stated the interpolation property for his p -adic L -function only for characters of degree at most 2, which is also sufficient for our purposes here.)

³For the main purposes of this article, ϵ shall be the trivial character. Despite this fact, we still chose to include it in our notation to have an easier comparison with Seveso’s notation in [37].

In particular, for the choice $\psi = \epsilon$ and in the neighborhood $k/2 + p\mathbb{Z}_p$ of $k/2$, Seveso's notation compares to those of [4, 27] via

$$\begin{aligned} L_p(\mathbf{f}_x, \epsilon, s) &= L_p(\mathbf{f}_x, \alpha(x), \epsilon\omega^{k/2-1}, s-1) \\ &= L_{p, \alpha(x)}(\mathbf{f}_x, \epsilon\omega^{k/2}, s). \end{aligned}$$

Note therefore that the interpolation property (7.1) reads

$$(7.2) \quad L_p(\mathbf{f}, \epsilon, k_x, s) = \lambda(x) L_{p, \alpha(x)}(\mathbf{f}_x, \epsilon\omega^{k/2}, s)$$

for s in the neighborhood $k/2 + p\mathbb{Z}_p$.

Let b_f^* be a canonical basis of $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{st}}(V_f^*)$ defined by Kato in [22, Theorem 12.5] and let

$$[\cdot, \cdot]_{V_f} : \mathbf{D}_{\mathrm{st}}(V_f) \times \mathbf{D}_{\mathrm{st}}(V_f) \longrightarrow E$$

denote the canonical pairing. It follows from the work of Kato [22, Theorem 16.2] (see also [4, Theorem 4.3.2]) that there exists an element $[z_{f, \mathrm{Iw}}^{\mathrm{BK}}] \in H_{\mathrm{Iw}}^1(G_{\mathbb{Q}, S}, V_f)$ such that

$$(7.3) \quad L_p([z_{f, \mathrm{Iw}}^{\mathrm{BK}}], s) = L_{p, \alpha}(f, \omega^{k/2}, s + k/2) [e_\alpha, b_f^*].$$

Hansen and Wang in [19, 43] recently constructed an element $\mathfrak{Z}_f^{\mathrm{BK}} \in H_{\mathrm{Iw}}^1(G_{\mathbb{Q}, S}, V_f)$ interpolating Kato's zeta elements⁴. Let

$$L_p(\mathfrak{Z}_f^{\mathrm{BK}}, \kappa, s) = \mathcal{A}(\mathfrak{Log}_f(\mathfrak{Z}_f^{\mathrm{BK}}))(\kappa, -s).$$

Then at each classical point x of weight $k_x \in \mathbb{Z}_{\geq 2} \cap \overline{B}(k, p^{-r})$ we have

$$(7.4) \quad L_p(\mathfrak{Z}_f^{\mathrm{BK}}, k_x, s) = c(x) L_{p, \alpha(x)}(\mathbf{f}_x, \omega^{k/2}, s + k/2)$$

for some non-zero constant $c(x) \in E_x$ (see the proof of [19, Proposition 4.2.2]). The following proposition follows from the work of Hansen and Wang.

Proposition 7.2 (Hansen, Wang). *We have*

$$L_p(\mathfrak{Z}_f^{\mathrm{BK}}, \kappa, s) \Big|_{s=\frac{\kappa-k}{2}} = a(\kappa) \cdot L_p(\mathbf{f}, \mathbf{1}, \kappa, \kappa/2)$$

where $a(\kappa)$ is analytic and non-vanishing in a neighborhood of $\kappa = k$ and $\mathbf{1}$ is the trivial character.

Proof. We define

$$\mathrm{Err}(\kappa, s) := \frac{L_p(\mathfrak{Z}_f^{\mathrm{BK}}, \kappa, s)}{L_p(\mathbf{f}, \mathbf{1}, \kappa, s + k/2)}.$$

Taking into account (7.2) together with (7.4) and relying on Rohrlich's non-vanishing result to eliminate common terms we see that

$$\mathrm{Err}(k_x, s) = \frac{c(x)}{\lambda(x)}$$

for every classical $x \in U$ of non-critical slope. Since such x are dense in U , it follows that $\mathrm{Err}(\kappa, s) = \mathrm{Err}(\kappa)$ does not depend on s . Furthermore, it is meromorphic and non-vanishing at every classical point of non-critical slope; in particular at k . The proof is complete on choosing $a(\kappa) = \mathrm{Err}(\kappa)$. \square

⁴Our $\mathfrak{Z}^{\mathrm{BK}}$ here coincides with $\mathfrak{Z}_1(k/2)$ of Hansen's paper.

Remark 7.3. It would be desirable to give a proof of Proposition 7.2 only working on the central critical line $s = \frac{\kappa-k}{2}$ (and only with the function $\mathfrak{L}_p(\kappa)$ on this line). This would have been possible if we knew that the ratios $c(x)/\lambda(x)$ that appear in the proof of Proposition 7.2 interpolate into a differentiable function at k . This however does not seem obvious, for example, because there is no a priori reason that the interpolation factors $\lambda(x)$ vary even continuously.

On the other hand, if one works with an *optimal* version of the interpolated Beilinson–Kato elements $[z_f^{\text{BK}}]$, one may in fact prove a more precise version of Proposition 7.2. As we are content with the conclusion of Proposition 7.2, we will not dwell on this matter here.

Let ϵ be an auxiliary quadratic or trivial character verifying

$$(7.5) \quad \epsilon(-N) = -\varepsilon_f \quad \text{and} \quad \epsilon(p) = 1.$$

Let L be an algebraic number field containing all a_n . Let $E \supset \mathbb{Q}_{p^2}$ denote its completion at a fixed (arbitrary) prime above p . Let the $\mathcal{M}_{k-2}/\mathbb{Q}$ denote the Iovita-Spiess Chow motive of weight k modular forms (whose p -adic realisation affords representations associated to cusp forms which are new at pN^-). There exists a map (via Faltings' comparison theorem and Coleman)

$$\log \Phi^{\text{AJ}} : \text{CH}^{k/2}(\mathcal{M}_{k-2} \otimes H) \longrightarrow M_k(\Gamma, E)^*,$$

(which is essentially identical to the p -adic Abel-Jacobi-map) where Γ is a certain ‘congruence subgroup’ of a suitably chosen quaternion algebra⁵, H is a certain extension of K and $M_k(\Gamma, E)^*$ is the dual of the space of rigid analytic modular forms for Γ . Let

$$\mathfrak{L}_p(\kappa) = L_p(\mathbf{f}, \epsilon, \kappa, s)|_{s=\kappa/2}$$

denote the restriction of $L_p(\mathbf{f}, \epsilon, \kappa, s)$ on the central critical line. The following result is [37, Theorem 6.1].

Theorem 7.4 (Seveso).

- i) *The p -adic L -function $\mathfrak{L}_p(\kappa)$ vanishes at $\kappa = k$ at least of order 2.*
- ii) *There exists a Heegner cycle*

$$y^\epsilon \in \text{CH}^{k/2}(\mathcal{M}_{k-2} \otimes H)$$

in the Chow group of codimension- $k/2$ cycles on and an element $t_f \in L^\times$ such that

$$\left. \frac{d^2 \mathfrak{L}_p}{d\kappa^2} \right|_{\kappa=k} = t_f \cdot \left(\log \Phi^{\text{AJ}}(y^\epsilon)|_{f^{\text{rig}}} \right)^2.$$

Here $f^{\text{rig}} \in M_k(\Gamma, E)$ is the rigid analytic modular form associated (via the Cerednik-Drinfeld uniformization) to the Jacquet-Langlands correspondent of f .

When the sign of the functional equation is -1 , the trivial character verifies (7.5). We shall eventually place ourselves in a situation where $r_{\text{an}}(f) = 1$ and we will take ϵ to be the trivial character.

⁵All this is made explicit in [37, Section 5.3.2]; see also [38].

7.2. Exceptional zeros of p -non-ordinary p -adic L -functions. Our goal in this section is to give a proof of Theorem A of the introduction. In the remainder of this paper, we assume that $f = \sum_{n=1}^{\infty} a_n q^n$ is an elliptic newform of even weight k for $\Gamma_0(Np)$ such that $a_p = p^{k/2-1}$ and $r_{\text{an}}(f) > 0$. Let $[z_f^{\text{BK}}] = \text{pr}_0([z_{f,\text{Iw}}^{\text{BK}}]) \in H^1(G_{\mathbb{Q},S}, V_f)$ denote the Beilinson-Kato element at the ground level. Kato's explicit reciprocity law and our hypothesis on $r_{\text{an}}(f)$ imply that $[z_f^{\text{BK}}] \in H^1(\mathbb{Q}, V_f)$. We denote by $[\mathfrak{z}_f^{\text{BK}}] \in \tilde{H}_f^1(V_f)$ the canonical lift of $[z_f^{\text{BK}}]$ with respect to the splitting spl of Proposition 4.2.

Theorem 7.5. *We have,*

$$\frac{\left(1 - \frac{1}{p}\right) \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle \cdot \langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle}{2 \cdot \Gamma(k/2 - 1)} \cdot \frac{d^2}{ds^2} L_{p,\alpha}(f, \omega^{k/2}, s) \Big|_{s=k/2} = \frac{\Re_{\mathfrak{g}_{\mathfrak{h}_p}}(\partial_0(d_{\tilde{\delta}}), [\mathfrak{z}_f^{\text{BK}}])}{[e_{\alpha}, b_f^*]_{V_f}}.$$

Proof. Recall that $\psi_k(\mathfrak{z}_f^{\text{BK}}) = c \cdot [z_{f,\text{Iw}}^{\text{BK}}]$ for some $c \neq 0$. Set $\mathfrak{X}_f^{\text{BK}} = c^{-1} \cdot \mathfrak{z}_f^{\text{BK}}$. It follows from Theorem 5.1(i) together with the defining properties of the classes $\mathfrak{z}_f^{\text{BK}}$ and $[z_{f,\text{Iw}}^{\text{BK}}]$ that

$$(7.6) \quad L_p(\mathfrak{X}_f^{\text{BK}}, k, s) = L_p([z_{f,\text{Iw}}^{\text{BK}}], s) = L_{p,\alpha}(f, \omega^{k/2}, s + k/2) [e_{\alpha}, b_f^*].$$

Set $\Omega_p := \left(1 - \frac{1}{p}\right) \Gamma\left(\frac{k}{2} - 1\right)^{-1} \langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle \cdot \langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle$, so that we have

$$\begin{aligned} \Omega_p \cdot L_p(\mathfrak{X}_f^{\text{BK}}, k, s) &\equiv \mathcal{A} \circ \det \begin{pmatrix} \mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), \partial_0(d_{\tilde{\delta}})) & \mathbb{H}_f(\partial_0(d_{\tilde{\delta}}), [\mathfrak{z}_f^{\text{BK}}]) \\ \mathbb{H}_f([\mathfrak{z}_f^{\text{BK}}], \partial_0(d_{\tilde{\delta}})) & \mathbb{H}_f([\mathfrak{z}_f^{\text{BK}}], [\mathfrak{z}_f^{\text{BK}}]) \end{pmatrix} \Big|_{\kappa=k} \\ &\equiv \det \begin{pmatrix} -\mathfrak{h}_p(\partial_0(d_{\tilde{\delta}}), \partial_0(d_{\tilde{\delta}})) & -\mathfrak{h}_p(\partial_0(d_{\tilde{\delta}}), [\mathfrak{z}_f^{\text{BK}}]) \\ -\mathfrak{h}_p([\mathfrak{z}_f^{\text{BK}}], \partial_0(d_{\tilde{\delta}})) & -\mathfrak{h}_p([\mathfrak{z}_f^{\text{BK}}], [\mathfrak{z}_f^{\text{BK}}]) \end{pmatrix} \cdot s^2 \pmod{s^3} \\ &= \Re_{\mathfrak{g}_{\mathfrak{h}_p}}(\partial_0(d_{\tilde{\delta}}), [\mathfrak{z}_f^{\text{BK}}]), \end{aligned}$$

where the first equality follows from Theorem 6.2 and the second from (4.11). The proof now follows combining this calculation with (7.6). \square

Corollary 7.6. *Assume that the Fontaine–Mazur $\mathcal{L}_{\text{FM}}(f)$ does not vanish. Then*

$$\begin{aligned} \left(1 - \frac{1}{p}\right) \frac{[e_{\alpha}, b_f^*]_{V_f}}{2\Gamma(k/2 - 1)} \langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle \cdot \frac{d^2}{ds^2} L_{p,\alpha}(f, \omega^{k/2}, s) \Big|_{s=k/2} = \\ \mathcal{L}_{\text{FM}}(f) \cdot h^{\text{Nek}}([z_f^{\text{BK}}], [z_f^{\text{BK}}]), \end{aligned}$$

where h^{Nek} denotes Nekovář's p -adic height in [30] associated to the splitting of the Hodge filtration of $\mathbb{D}_{\text{st}}(V_f)$ induced by D .

Proof. The proof is formal and repeats verbatim the arguments of Nekovář; see the proof of [31, Proposition 11.4.9]. By [6, Proposition 20], the pairing h^{Nek} coincides with the pairing $h_{V_f, D}^{\text{spl}}$ constructed in [6]. Theorem 11 and Lemma 10 of [6] together give

$$h^{\text{Nek}}([z_f^{\text{BK}}], [z_f^{\text{BK}}]) = \mathfrak{h}_p([z_f^{\text{BK}}], [z_f^{\text{BK}}]) + \frac{\langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle^2}{\langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle}.$$

Using Theorem 4.10, we have

$$\Re_{\mathfrak{h}_p}(\partial_0(d_{\tilde{\delta}}), [z_f^{\text{BK}}]) = -\mathfrak{h}_p([z_f^{\text{BK}}], [z_f^{\text{BK}}]) \langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle - \langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle^2.$$

Therefore

$$\frac{\Re_{\mathfrak{h}_p}(\partial_0(d_{\tilde{\delta}}), [z_f^{\text{BK}}])}{\langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle} = -h^{\text{Nek}}([z_f^{\text{BK}}], [z_f^{\text{BK}}]) \cdot \frac{\langle \Psi_2, \partial_0(d_{\tilde{\delta}}) \rangle}{\langle \Psi_1, \partial_0(d_{\tilde{\delta}}) \rangle} = \mathcal{L}_{\text{FM}}(f) \cdot h^{\text{Nek}}([z_f^{\text{BK}}], [z_f^{\text{BK}}]).$$

Now the Corollary follows from Theorem 7.5. \square

Remark 7.7. Theorem 7.5 can be deduced from the existence of Beilinson–Kato classes $[z_{f,\text{Iw}}^{\text{BK}}]$ and the one variable analog of Theorem 6.2, without any appeal to deformations of V_f besides the cyclotomic deformation \overline{V}_f .

7.3. On the conjecture of Perrin-Riou. We keep notation and conventions of the previous Subsection. Assume, in addition, that $\varepsilon_f = -1$ and ϵ is chosen as the trivial character. Note that the condition (7.5) holds.

Theorem 7.8. *We have the following comparison between the Heegner cycle and the Beilinson-Kato element:*

$$t_f \cdot \left(\log \Phi^{\text{AJ}}(y^\epsilon) \Big|_{f^{\text{rig}}} \right)^2 \langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle = a(k) \left(1 - \frac{1}{p} \right)^{-1} \Gamma \left(\frac{k}{2} - 1 \right) \cdot \frac{\langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle^2}{2}.$$

Proof. Recall that $\psi_k(\mathfrak{X}_f^{\text{BK}}) = [z_{f,\text{Iw}}^{\text{BK}}]$ and $\text{pr}_{\gamma,\kappa}(\mathfrak{X}_f^{\text{BK}}) = [z_f^{\text{BK}}]$. We have

$$(7.7) \quad t_f \cdot \left(\log \Phi^{\text{AJ}}(y^\epsilon) \Big|_{f^{\text{rig}}} \right)^2 \cdot \langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle = \frac{d^2 \mathfrak{L}_p}{d\kappa^2} \Big|_{\kappa=k}$$

$$(7.8) \quad = a(k) \cdot \frac{d^2}{d\kappa^2} \mathfrak{L}_p(\mathfrak{X}_f^{\text{BK}}, \kappa) \Big|_{\kappa=k}$$

$$(7.9) \quad = a(k) \left(1 - \frac{1}{p} \right)^{-1} \Gamma \left(\frac{k}{2} - 1 \right) \cdot \frac{\langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle^2}{2}$$

where (7.7) is Theorem 7.4, (7.8) is Theorem 7.2 and finally, (7.9) is Theorem 6.3. \square

The following theorem gives a partial answer to a question of Perrin-Riou.

Theorem 7.9. *Suppose that $\varepsilon_f = -1$ and ϵ is chosen as the trivial character. Assume that $\log \Phi^{\text{AJ}}(y^\epsilon) \Big|_{f^{\text{rig}}}$ is non-vanishing. Then the restriction of the Beilinson–Kato class $\text{res}_p([z_f^{\text{BK}}])$ at the prime p does not vanish.*

Proof of Theorem 7.9. It follows from Theorem 7.4(ii) that $\text{ord}_{\kappa=k} \mathfrak{L}_p(\kappa) = 2$. Proposition 6.4 in turn implies that $\text{res}_p([z_f^{\text{BK}}]) \neq 0$, as desired. \square

Corollary 7.10. *Suppose that the p -adic Abel–Jacobi image $\log \Phi^{\text{AJ}}(y^\epsilon) \Big|_{f^{\text{rig}}}$ of the Heegner cycle y^ϵ is non-trivial. Assume further that the (cyclotomic) p -adic height pairing \mathfrak{h}_p is non-degenerate. Then,*

$$\text{ord}_{s=k/2} L_p(f, s) = 2.$$

Proof. It follows from Theorem 7.9 that $\langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle$ and therefore also Ω_p is non-zero. Theorem 7.9 also shows that the Beilinson-Kato class $[z_f^{\text{BK}}] \in H_f^1(V_f)$ is non-trivial, so that the pair $\{\partial_0(d_{\tilde{g}}), [z_f^{\text{BK}}]\}$ is linearly independent. The proof follows by our assumption that \mathfrak{h}_p is non-degenerate. \square

Remark 7.11. If $\text{res}_p([z_f^{\text{BK}}]) \neq 0$, we infer using Theorem 7.8 that

$$t_f \cdot \left(\log \Phi^{\text{AJ}}(y^\epsilon) \Big|_{f^{\text{rig}}} \right)^2 = a(k) \left(1 - \frac{1}{p} \right)^{-1} \Gamma \left(\frac{k}{2} - 1 \right) \cdot \frac{\langle \Psi_2, \text{res}_p([z_f^{\text{BK}}]) \rangle}{2}.$$

Remark 7.12. When the order of vanishing $r_{\text{an}}(f)$ of the Hecke L -function $L(f, s)$ equals 1 (and only then) the hypothesis of Theorem 7.9 is expected to hold true. A suitable extension of Gross–Zagier–Zhang formula on Shimura Curves shows that the Heegner cycle y^ϵ is non-torsion iff $r_{\text{an}}(f) = 1$. The desired formula is available in the literature for weight two forms and for Heegner cycles on classical modular curves for higher weight forms by the celebrated works of Zhang [44, 45]. The experts seem to believe that the current technology would be sufficient to extend these results to our case of interest. Furthermore, the p -adic Abel–Jacobi map Φ^{AJ} is also expected to be injective in this set up. When the weight of the eigenform equals to 2, this follows from the finiteness of the Tate–Shafarevich group proved by Kolyvagin.

Remark 7.13. As we have already pointed out in Remark 7.12, a suitable Gross–Zagier–Zhang formula would show that the non-vanishing of the Abel–Jacobi image of the Heegner cycle y^ϵ is equivalent to asking that the Hecke L -function associated to f vanishes at $s = k/2$ to exact order 1. If this is the case, one may prove that $H_f^1(\mathbb{Q}, V_f)$ is one-dimensional and the requirement that \mathfrak{h}_p be non-degenerate is equivalent to asking that it is non-zero.

REFERENCES

- [1] J. Bellaïche, *Critical p -adic L -functions*, Invent. Math. **189** (2012), 1-60.
- [2] D. Benois, *Infinitesimal deformations and the ℓ -invariant*, (extra volume: Andrei A. Suslin's sixtieth birthday) Doc. Math. (2010), 5-31.
- [3] D. Benois, *A generalization of Greenberg's \mathcal{L} -invariant*, Amer. J. Math. **133** (2011), 1573-1632.
- [4] D. Benois, *Trivial zeros of p -adic L -functions at near central points*, J. Inst. Math. Jussieu **13** (2014), 561-598.
- [5] D. Benois, *Selmer complexes and p -adic Hodge theory*, in “Arithmetic and Geometry” London Mathematical Society Lecture Note Series **420**, 36-88.
- [6] D. Benois, *p -adic heights and p -adic Hodge theory*, preprint, 2014, available at <http://arxiv.org/abs/1412.7305>
- [7] D. Benois and L. Berger, *Théorie d'Iwasawa des représentations cristallines II*, Comment. Math. Helv. **83** (2008), 603-677.
- [8] L. Berger, *Représentations p -adiques et équations différentielles*, Invent. Math. **148** (2002), 219-284.
- [9] L. Berger *Bloch and Kato's exponential map: three explicit formulas*, Doc. Math., Extra Volume: Kazuya Kato's Fiftieth Birthday (2003), 99-129.
- [10] L. Berger, *Équations différentielles p -adiques et (φ, N) -modules filtrés*, In: Représentations p -adiques de groupes p -adiques I (L. Berger, P. Colmez, Ch. Breuil eds.), Astérisque **319** (2008), 13-38

- [11] M. Bertolini and H. Darmon, *Hida families and rational points on elliptic curves*, Invent. Math. **168** (2007), no. 2, 371-431.
- [12] K. Büyükboduk, *On Nekovář's heights, exceptional zeros and a conjecture of Mazur-Tate-Teitelbaum*, IMRN, to appear.
- [13] H. Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Composition Math. **59** (1986), 151-230.
- [14] P. Colmez, *La conjecture de Birch et Swinnerton-Dyer p -adique*, Séminaire Bourbaki 2002/03, Astérisque **294** (2004), 251-319.
- [15] P. Colmez, *Représentations triangulines de dimension 2*, In: Représentations p -adiques de groupes p -adiques I (L. Berger, P. Colmez, Ch. Breuil eds.), Astérisque **319** (2008), 213-258.
- [16] R. F. Coleman, *p -adic Banach spaces and families of modular forms*, Invent. Math. **127**, 417-479 (1997).
- [17] R. F. Coleman and A. Iovita, *Hidden structures on semistable curves*, Astérisque **331** (2010), 179-254.
- [18] P. Colmez, *Invariant \mathcal{L} et dérivées de valeurs propres de Frobenius*, Astérisque **331** (2010), 13-28.
- [19] D. Hansen, *Iwasawa theory of overconvergent modular forms, I: Critical p -adic L -functions*, 30 pages, preprint, 2015, available at <http://arxiv.org/abs/1508.03982>
- [20] L. Herr, *Sur la cohomologie galoisienne des corps p -adiques*, Bull. Soc. Math. France **126** (1998), 563-600.
- [21] A. Iovita and M. Spiess, *Derivatives of p -adic L -functions, Heegner cycles and monodromy modules attached to modular forms*, Invent. Math., **154** (2003), 333-384.
- [22] K. Kato, *p -adic Hodge theory and values of zeta functions of modular forms*, in: Cohomologies p -adiques et applications arithmétiques. III, Astérisque **295**, (2004), 117-290.
- [23] K. Kedlaya, *A p -adic monodromy theorem*, Ann. of Math., **160** (2004), 93-184.
- [24] K. Kedlaya, J. Pottharst, L. Xiao, *Cohomology of arithmetic families of (φ, Γ) -modules*, Journal of the Amer. Math. Soc. **27** (2014), 1043-1115.
- [25] R. Liu, *Cohomology and Duality for (φ, Γ) -modules over the Robba ring*, Int. Math. Research Notices (2007), no. 3, 32 pages.
- [26] R. Liu, *Triangulation of refined families*, 56 pages, preprint, 2014, available at <http://arxiv.org/abs/1202.2188>
- [27] B. Mazur, J. Tate, J. Teitelbaum, *On p -adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. math. **84** 1-48 (1986).
- [28] K. Nakamura, *Iwasawa theory of de Rham (φ, Γ) -modules over the Robba ring*, J. Inst. Math. Jussieu, **13** (2014), 65-118.
- [29] K. Nakamura, *A generalization of Kato's local ε -conjecture for (φ, Γ) -modules over the Robba ring*, preprint available at <http://arxiv.org/abs/1305.0880>
- [30] J. Nekovář, *On p -adic height pairings*, In: Séminaire de Théorie des Nombres, Paris, 1990/91 (S. David, ed.), Progress in Mathematics 108, Birkhäuser Boston, 1993, pp. 127-202.
- [31] J. Nekovář, *Selmer complexes*, Astérisque **310** (2006), 559 pages.
- [32] A. Panchishkin, *Two-variable L -functions associated to eigenfamilies of positive slope*, Invent. Math., **154** (2003), 551-615.
- [33] B. Perrin-Riou, *Fonctions L p -adiques d'une courbe elliptique et points rationnels*, Ann. Inst. Fourier (Grenoble) (1993), **43**, issue 4, pp 945-995.
- [34] B. Perrin-Riou, *Théorie d'Iwasawa des représentations p -adiques sur un corps local*, Invent. Math. **115** (1994), 137-185.
- [35] J. Pottharst, *Analytic families of finite slope Selmer groups*, Algebra and Number Theory **7** (2013), 1571-1611.
- [36] J. Pottharst, *Cyclotomic Iwasawa theory of motives*, (2011), Preprint available on <http://math.bu.edu/people/potthars/>.
- [37] M.A. Seveso, *p -adic L -functions and the rationality of Darmon cycles*, Can. J. Math. **64**, No. 5, 1122-1181 (2012).
- [38] M.A. Seveso, *Heegner cycles and derivatives of p -adic L -functions*, J. Reine Angew. Math. **686** (2014), 111-148.

- [39] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press (1971).
- [40] G. Stevens, *Coleman's \mathcal{L} -invariant and families of modular forms*, Astérisque **311** (2010), 1-12.
- [41] R. Venerucci, *p-adic regulators and p-adic families of modular forms*, Ph.D. Thesis (2013), Università degli studi di Milano.
- [42] R. Venerucci, *Exceptional zero formulae and a conjecture of Perrin-Riou*, Invent. Math., to appear.
- [43] S. Wang, *Le système d'Euler de Kato en famille (II)* , 36 pages, preprint, 2013, available at <http://arXiv:1312.6428v2>
- [44] S. Zhang, *Heights of Heegner cycles and derivatives of L-series*, Invent. Math. **130**, No.1, 99-152 (1997).
- [45] S. Zhang, *Heights of Heegner points on Shimura curves*, Ann. Math. (2) **153**, No. 1, 27-147 (2001).

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